Resonances in Dynamical Systems

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Introduction to Dynamical Systems

Roughly speaking, a dynamical system is a system that evolves (changes) in time according to some rules. This includes the position of the planets, moons, and stars, flows of fluids, and the movement of a wave or particle.

The rules that govern these systems might be complex and unknown. We use physical priniciples to derive a set of rules.

Definition 1.1 (Dynamical System)**.** A dynamical system is a 3-tuple of

- a state space X,
- a set of times T,
- a function $\Phi\colon T \times X \to X$.

Usually, one also adds a semigroup structure both to T and Φ.

We will use $X = \mathbb{R}^n$ or $X = M$ a euclidean manifold (or X a function space). For continuous systems, we have $T = \mathbb{R}$ or $T = \mathbb{R}_+ = [0, \infty)$. For discrete systems, we have $T = \mathbb{Z}$ or $T = \mathbb{N}_0$.

$\Phi(t, x_0)$ gives the state of the system at time $t \in T$ if the system was initially (that is say at $t = 0$) in the state $x_0 \in X$.

This definition is incredibly abstract and much too general for the purpose of this course. In practice, we will think of a continuous dynamical system to be the flow of an autonomous ordinary differential equation

$$\dot{x} = f(x). \tag{1.1}$$

Assuming (1.1) has global-in-time unique solutions (for example if f is Lipschitz), we may define the flow via

$$\Phi(t, x_0) = x(t), \quad \text{where } x \text{ is the unique solution to } \begin{cases} \dot{x} = f(x), \\ x(0) = x_0. \end{cases}$$

Note that Φ is a semigroup: it holds for all $x_0 \in X$ and $t, s \in \mathbb{R}$

(1) $\Phi(0, x_0) = x_0$,

(2) $\Phi(t, \Phi(s, x_0)) = \Phi(s, \Phi(t, x_0)) = \Phi(t + s, x_0)$.

Observe in particular that Φ is invertible. In fact, if f is smooth, Φ is a continuous family of diffeomorphisms of the state space X.

A discrete dynamical system might come from sampling a continuous time system. Thereforem, choosing a time-step size t_0, we may define the map

$$F(x_0) = \Phi(t_0, x_0).$$

Then we obtain a discrete dynamical system

$$\widetilde{\Phi}\colon \mathbb{Z} \times X \to X : \tilde{\Phi}(n, x_0) = \Phi(nt_0, x_0) = F^n(x_0).$$

Here and in the future, we denote by F^n the n-th iterate of F[1]. We will only consider discrete dynamical systems to be of this form, i.e. to come from a homeomorphism[2] $F\colon X \to X$.

[1] That is

$$F^n = \underbrace{F \circ \cdots \circ F}_{n \text{ times}}.$$

[2] Or very soon from a diffeomorphism in a specific regularity class.

Example 1.2 (Continuous systems)**.**

(a) Simple harmonic oscillator: consider a mass m hanging of a spring. If x denotes the displacement of a spring from its equilibrium position and using Hooke's law[3], then the dynamical behaviour of the spring is given by

$$m\ddot{x} = -kx.$$

This equation can be transformed into a first-order autonomous ODE of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ mv \end{pmatrix} = \begin{pmatrix} v \\ -kx \end{pmatrix}$$

[3] Hooke's law: the restoring force of a spring is proportional to the displacement

$$F_{\text{Hooke}} = -kx,$$

k the spring constant.

(b) Two-body problem: the motion of two bodies through physical space $\mathbb{R}^3$ with masses m_1, m_2 considering gravitational pull is given by

$$m_1\ddot{x}_1 = F_{12}(x_1 - x_2) = \tfrac{gm_1m_2(x_2-x_1)}{|x_1-x_2|^3}$$
$$m_2\ddot{x}_2 = F_{21}(x_1 - x_2) = -F_{12}(x_1 - x_2).$$

Here $g > 0$ denotes the gravitational constant.

(c) Hamiltonian systems: there is an underlying structure for (a) and (b). If $H = H(q,p)\colon X \times TX \to \mathbb{R}$ is a function, we define the correspoding *Hamiltonian system* via

$$\dot{q} = \tfrac{\partial H}{\partial q}$$
$$\dot{p} = -\tfrac{\partial H}{\partial p}.$$

Here, TX is the abstract tangent space of X. So if $X = \mathbb{R}^n$, then we will identify $TX \cong X$.

We interpret q to be the position and p as the momentum of our moving body.

For (a), we have $X = TX = \mathbb{R}$ and the Hamiltonian is given by

$$H(q,p) = \frac{k^2}{2} - \frac{kq^2}{2m}.$$

For (b), we have $X = TX = \mathbb{R}^3 \times \mathbb{R}^3$ and we have

$$H(q_1, q_2, p_1, p_2) = K - U,$$
$$K(p_1, p_2) = \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2},$$
$$U(q_1, q_2) = \frac{gm_1m_2}{|q_1 - q_2|}.$$

K denotes the *kinetic energy* of the system and U denotes the *potential energy* of the system.

(d) Infinite dimensional dynamical systems: such as the wave equation

$$\partial_t^2 u - \Delta u = 0.$$

Example 1.3 (Discrete dynamical systems). (a) Any diffeomorphism of a euclidean manifold M, $f\colon M \to M$ gives rise to a dynamical system

$$\Phi\colon \mathbb{Z} \times M \to M : \Phi(n, x_0) = f^n(x_0).$$

(b) Rotations of the circle: denote by $S^1 = \{e^{2\pi i\theta} : \theta \in [0,1]\}$ the circle and let $M = S^1$. For $\alpha \in [0,1)$, consider the diffeomorphism $R_\alpha\colon S^1 \to S^1$ given by rotation with angle $2\pi\alpha$, that is

$$R_\alpha(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}.$$

Observe that then

$$R_\alpha(e^{2\pi i\theta})^n = e^{2\pi i(\theta+n\alpha)} = R_{n\alpha}(e^{2\pi i\theta}).$$

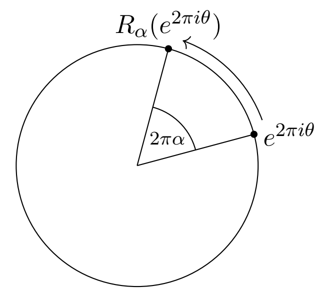

Fig. 1.1: Rotation with angle $2\pi\alpha$

(c) The Chirikov standard map: denote by $\mathbb{T}^n$ the n-dimensional torus[4] and consider the diffeomorphism

$$f\colon \mathbb{T}^2 \to \mathbb{T}^2 : f(e^{2\pi i\theta}, e^{2\pi ip}) = (e^{2\pi i(\theta+p+K\sin\theta)}, e^{2\pi i(p+K\sin\theta)}).$$

[4] $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$.

We will study the rotations on the circle in more detail in Chapter 2.

A general dynamical system can have very different kinds of behaviour. There might be order or chaos. We collect different notions of order.

Definition 1.4. Given a dynamical system (X, T, Φ), we define

(i) We define the *orbit* starting at $x_0 \in X$ by

$$\gamma_{x_0} = \{\Phi(t, x_0) : t \in T\}.$$

(ii) We call $x_0 \in X$ a *stationary point*[5] if

$$\gamma_{x_0} = \{x_0\}.$$

[5] Observe that for an autonomous ODE $\dot{x} = f(x)$, a point is a stationary point if and only if $f(x_0) = 0$. For a discrete system given by a homeomorphism $F\colon X \to X$, a point is stationary if and only if it is a fixed point $F(x_0) = x_0$.

(iii) An orbit γ_{x_0} is called *periodic* if there is $t \neq 0$ such that

$$\Phi(t, x_0) = x_0.$$

(iv) An orbit γ_{x_0} is called *quasi-periodic* if $t \mapsto \Phi(t, x_0)$ is a quasi-periodic function[6], that is if

$$\Phi(t, x_0) = f(\omega_1 t, \dots, \omega_n t),$$

where $f\colon T^n \to X$, for some $n \in \mathbb{N}$, is periodic in each component and $\omega_1, \dots, \omega_n \in \mathbb{R}$ are rationally independent.

[6] For example the function

$$t \mapsto \sin(at) + \sin(bt)$$

with a and b rationally independent is quasi-periodic.

2

KAM theory for diffeomorphisms of the circle

In this chapter, we will discuss the KAM theory for diffeomorphisms of the circle. We begin by analysing the dynamics of the rotation map. Before we study small analytic perturbations of the rotations, we need to define the rotation number and we will prove Denjoy's theorem.

Before we start, we need to fix some notation: consider the circle $S^1 = \{e^{2\pi i\theta} \in \mathbb{C} : \theta \in [0,1)\} \cong \mathbb{R}/\mathbb{Z}$[1]. Then $\mathbb{R}$ forms a *covering space* of S^1 with cover given by

[1] We will relatively freely identify both constructions.

$$\pi\colon \mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}.$$

In particular, $\pi(x+z) = \pi(x)$ for all $z \in \mathbb{Z}$.

Definition 2.1. Let $f\colon S^1 \to S^1$ a continuous map of the circle. Then $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is called a *lift* of f to R if

$$\pi \circ \tilde{f} = f \circ \pi.$$

Lemma 2.2. *Every continuous map $f\colon S^1 \to S^1$ has a lift $\tilde{f}$ to $\mathbb{R}$.*

Proof. Exercise. □

Remark 2.3.
1) If $f\colon S^1 \to S^1$ is a homeomorphism of the circle, then $\tilde{f}$ is strictly monotone. If f is orientation-preserving, then $\tilde{f}$ is increasing. If f is orientation-reversing, then $\tilde{f}$ decreases.
2) If $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is a lift of $f\colon S^1 \to S^1$, then so is $\tilde{f} + m$ for any $m \in \mathbb{Z}$. Vice versa, all lifts differ only by a translation by $m \in \mathbb{Z}$.
3) For every lift it holds
$$\tilde{f}(x+1) = \tilde{f}(x) + d.$$
If $f\colon S^1 \to S^1$ is a homeomorphism, then $d \in \{\pm 1\}$.[2] Here, $d = 1$ if and only if f is orientation-preserving. d is called the *degree* of f and $x \mapsto F(x) - dx$ is periodic with period 1.

[2] The opposite direction is false in general.

4) The map $\tilde{f}(x) - x$ is periodic with period 1.

Exercise. Let $f\colon S^1 \to S^1$ be an orientation-reversing homeomorphism. Show that f has exactly two fixed points.

Example 2.4. Let $f\colon S^1 \to S^1 : f(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}$ the rotation by angle $\alpha \in [0,1)$. Then $\tilde{f}(x) = x + \alpha$ is a lift.

1 *The dynamics of the rotation map*

Definition 2.5. For $\alpha \in [0,1)$ we define the map

$$R_\alpha \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} : R_\alpha(x) = (x+\alpha) \bmod \mathbb{Z}$$

as the rotation of the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

Observe that $R_\alpha^n(x_0) = x_0 + n\alpha \bmod \mathbb{Z}$ for any $n \in \mathbb{Z}$. We will now study the dynamics of the rotation map.

Proposition 2.6. *Let $\alpha \in [0,1)$.*

(1) If $\alpha \in \mathbb{Q}$ is rational, then every orbit of R_α is periodic. If $\alpha = \frac{p}{q}$ with p, q coprime, then$q$ is the period of each orbit.

(2) If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is irrational, then every orbit is dense[3] in $\mathbb{R}/\mathbb{Z}$.

[3] Sometimes, dynamical systems with this property are called minimal.

In order to prove the density of orbits, we rely on a result from number theory.

Lemma 2.7 (Dirichlet's approximation theorem)**.** *Let $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$. Then there is a pair of integers $(p,q) \in \mathbb{Z} \times \mathbb{N}$ with $1 \le q \le N$ such that*

$$|\alpha q - p| < \frac{1}{N}.$$

Proof. [4] This can be proved via the pigeonhole principle. Consider the $N+1$ numbers $k\alpha$, $k = 0, \ldots, N$. They can be uniquely written as

$$k\alpha = m_k + x_k, \quad m_k \in \mathbb{Z},\ 0 \le x_k < 1.$$

Now the set $\{x_0, \ldots, x_N\} \subset [0,1)$ consists of $N+1$ numbers, hence by the pigeonhole principle there must be two numbers x_i and x_j with $i < j$ such that

$$|x_j - x_i| < \frac{1}{N}.$$

But now observe that

$$|(j-i)\alpha - (m_j - m_i)| = |j\alpha - m_j - (i\alpha - m_i)| = |x_j - x_i| < \frac{1}{N}.$$

This proves the theorem with the choice $q = (j-i) \in \mathbb{N}$ and $p = m_j - m_i \in \mathbb{Z}$. □

[4] There is a deep connection to continued fractions: for a number $\alpha \in \mathbb{R}$, we define

$$\alpha = \lfloor \alpha \rfloor + \tfrac{1}{\alpha_1} = a_0 + \tfrac{1}{\alpha_1}$$
$$\alpha_1 = \lfloor \alpha_1 \rfloor + \tfrac{1}{\alpha_2} = a_1 + \tfrac{1}{\alpha_2}$$
$$\vdots$$

then

$$\alpha = a_0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\vdots}{a_{n-1} + \cfrac{1}{\alpha_n}}}}}$$

and we call

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\vdots}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

the n-*th convergent* of α. Then the following results hold true:

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}.$$

And vice versa, if

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2},$$

then

$$\frac{p}{q} \in \left\{ \frac{p_n}{q_n}, \frac{p_{n+1}+p_n}{q_{n+1}+q_n}, \frac{p_{n+2}-p_{n+1}}{q_{n+2}-q_{n+1}} \right\}$$

Using that $1 \le q \le N$, the following corollary is an immediate consequence.

Corollary 2.8. *For any real number $\alpha \in \mathbb{R}$ there exist infinitely many pairs of integers $(p,q) \in \mathbb{Z} \times \mathbb{N}$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Now we are able to prove Proposition 2.6.

Proof. **Step 1: rational angles** If $\alpha = \frac{p}{q}$ and $x_0 \in \mathbb{R}/\mathbb{Z}$ is arbitrary, then

$$R_\alpha^q(x_0) = x_0 + q \cdot \frac{p}{q} \mod \mathbb{Z} = x_0.$$

Hence, γ_{x_0} is a periodic orbit. If p and q are coprime, then q is the smallest number such that $q \cdot \frac{p}{q} \in \mathbb{Z}$.

Step 2: irrational angles Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $x_0 \in [0, 1)$. We first prove:

Claim 1: if $m \neq n$, then $R_\alpha^m(x_0) \neq R_\alpha^n(x_0)$. Indeed, assume that $R_\alpha^m(x_0) = R_\alpha^n(x_0)$, then

$$x + n\alpha \mod \mathbb{Z} = x + m\alpha \mod \mathbb{Z},$$

which implies that

$$(m - n)\alpha \in \mathbb{Z}.$$

But since $\alpha \notin \mathbb{Q}$, this can only be satisfied if $m = n$.

Claim 2: the orbit is dense[5]. Therefore, let $\varepsilon > 0$. By Corollary 2.8 there is a pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $\frac{1}{q} < \varepsilon$ such that

$$|q\alpha - p| < \frac{1}{q} < \varepsilon.$$

But this readily implies that

$$|(R_\alpha^q(x_0) - x_0) \mod \mathbb{Z}| = |q\alpha - p| < \varepsilon.$$

Consider, for M large enough the set of points

$$\{x_0, R_\alpha^q(x_0), R_\alpha^{2q}(x_0), \dots, R_\alpha^{Mq}(x_0)\}.$$

Then this set breaks S^1 into M intervals of length smaller than ε. Hence, for every $x \in [0, 1)$ there must exist $k \in \{0, \dots, M\}$ such that

$$|x - R_\alpha^{kq}(x_0)| < \varepsilon,$$

which proves the density of the orbit. □

[5] One could also use Weyl's equidistribution criterion. This states that a sequence $(x_n)_{n \in \mathbb{N}_0}$ is equidistributed in [0, 1] if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i \ell x_n} = 0$$

holds for every $\ell \neq 0$.

Applying this to the sequence $x_n = R_\alpha^n(x_0)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ gives equidistribution. Actually, the dynamical system is ergodic: every measurable, invariant set is either a Lebesgue null set or of full Lebesgue measure.

2 *Rotation number and Denjoy's theorem*

In the remaining part of this chapter we will be concerned with the question: when does an arbitrary (orientation-preserving) homeomorphism (or diffeomorphism) behave like a rotation? To understand which rotation to choose, we introduce the rotation number.

Definition 2.9. Let $f : S^1 \to S^1$ an orientation-preserving homeomorphism and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ a lift of f and let $x_0 \in [0, 1)$. Then the *rotation number* of f is given by

$$\rho(f) = \Big[\lim_{|n| \to \infty} \frac{\tilde{f}^n(x_0) - x_0}{n}\Big] \mod \mathbb{Z}. \tag{2.1}$$

Theorem 2.10. *Let $f : S^1 \to S^1$ an orientation-preserving homeomorphism. Then the rotation number $\rho(f)$ exists and is independent of the choice of $x_0 \in [0, 1)$.*

Proof. **Step 1:** The rotation number is independent of the choice of x_0. Fix $x_0, y_0 \in [0,1)$. Let w.l.o.g. $x_0 < y_0$. Then it holds by Remark 2.3 that

$$\tilde{f}^n(y_0) - \tilde{f}^n(x_0) < \tilde{f}^n(y_0+1) - \tilde{f}^n(y_0) = 1.$$

Now we can estimate

$$\begin{aligned} \left| \frac{\tilde{f}^n(x_0) - x_0 - (\tilde{f}^n(y_0) - y_0)}{n} \right| &\le \frac{|\tilde{f}^n(x_0) - \tilde{f}^n(y_0)| + |x_0 - y_0|}{n} \\ &\le \frac{2}{n} \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Hence, the limit is independent of the choice of x_0.
Step 2: Existence of the limit. We will distinguish the cases: f has a periodic point and f has no periodic point.

If f has a periodic point with period $m \in \mathbb{N}$, this implies that there must be an element $z \in \mathbb{Z}$ and $x_0 \in [0,1]$ such that $\tilde{f}^m(x_0) = x_0 + z$. Hence, it must hold

$$\tilde{f}^{km}(x_0) = x_0 + kz, \quad k \in \mathbb{N}.$$

But from this, we conclude that

$$\lim_{k\to\infty} \frac{|\tilde{f}^{km}(x_0) - x_0|}{km} = \frac{kz}{km} = \frac{z}{m} \in \mathbb{Q}.$$

We need to show that the full sequence is converging. Therefore, write $n = km + r$, with $0 \le r < m$. Since $\tilde{f} - \mathrm{Id}$ is periodic and continuous, we find a number $M > 0$ such that

$$|\tilde{f}(x) - x| \le M$$

for all $x \in \mathbb{R}$ and we may conclude

$$\frac{|\tilde{f}^n(x_0) - x_0 - (\tilde{f}^{km}(x_0) - x_0)}{n} = \frac{|\tilde{f}^r(\tilde{f}^{km}(x_0)) - \tilde{f}^{km}(x_0)|}{n} \le \frac{M}{n} \xrightarrow[n\to\infty]{} 0.$$

If f has no periodic points, then we know that $\tilde{f}^n(x) - x \notin \mathbb{Z}$ for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Since $\tilde{f}^n - \mathrm{Id}$ is periodic and continuous, this means that there is $z \in \mathbb{Z}$ such that

$$z < \tilde{f}^n(x) - x < z + 1$$

for all $x \in \mathbb{R}$. Choosing $x = 0$, we find that

$$z < \tilde{f}^n(0) < z + 1$$

and choosing $x = \tilde{f}^n(0)$ und using monotonicity, we find

$$\tilde{f}^n(0) + z = \tilde{f}^n(z) \le \tilde{f}^{2n}(0) \le \tilde{f}^n(z+1) = \tilde{f}^n(0) + z + 1.$$

Hence,

$$z < \tilde{f}^{2n}(0) - \tilde{f}^n(0) < z + 1$$

and by induction

$$z < \tilde{f}^{kn}(0) - \tilde{f}^{(k-1)n}(0) < z + 1$$

holds for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Summing over k, we find

$$kz < \tilde{f}^{kn}(0) < k(z+1)$$

for all $k, n \in \mathbb{N}$, or dividing by kn, we have

$$\frac{z}{n} < \frac{\tilde{f}^{kn}(0) - 0}{kn} < \frac{z+1}{n}$$

for every $k, n \in \mathbb{N}$. In particular, it holds

$$\left| \frac{\tilde{f}^{kn}(0) - 0}{kn} - \frac{\tilde{f}^{n}(0) - 0}{n} \right| < \frac{1}{n}. \tag{2.2}$$

But now, we obtain by (2.2) that

$$\begin{aligned} \left| \frac{\tilde{f}^{n}(0) - 0}{n} - \frac{\tilde{f}^{m}(0) - 0}{m} \right| &\leq \left| \frac{\tilde{f}^{n}(0)}{n} - \frac{\tilde{f}^{mn}(0) - 0}{nm} \right| + \left| \frac{\tilde{f}^{mn}(0) - 0}{mn} - \frac{\tilde{f}^{m}(0)}{m} \right| \\ &\leq \frac{1}{n} + \frac{1}{m}. \end{aligned}$$

This proves that the sequence $\left(\frac{\tilde{f}^{n}(0)}{n}\right)_n$ is Cauchy and so it has a limit. ☐

Example 2.11. For $\alpha \in [0,1)$ it is $\rho(R_\alpha) = \alpha$.

Lemma 2.12. *Let $f \colon S^1 \to S^1$ an orientation-preserving homeomorphism.*

1) *If $h \colon S^1 \to S^1$ is an orientation-preserving homeomorphism, then*

$$\rho(f) = \rho(h^{-1} \circ f \circ h).$$

2) *If $m \in \mathbb{N}$, then*

$$\rho(f^m) = m\rho(f) \mod \mathbb{Z}.$$

3) *$\rho(f) \in \mathbb{Q}$ is rational if and only if f has a periodic orbit.*

Proof. **2)** Note that

$$\frac{(\tilde{f}^m)^n(x_0) - x_0}{n} = \frac{\tilde{f}^{mn}(x_0) - x_0}{n} = m\frac{\tilde{f}^{mn}(x_0) - x_0}{mn}.$$

3) We have already seen that if we have a periodic orbit, then the rotation number is rational. Now assume that the rotation number is rational with $\rho(f) = \frac{p}{q} \in \mathbb{Q}$. We will show that we can construct an orbit of period q.

By 2), we have

$$\rho(f^q) = q\rho(f) \mod \mathbb{Z} = 0.$$

It hence suffices to show that if for a general homeomorphism we have $\rho(f) = 0$, then f has a stationary point. We prove the contraposition: assume f does not have a stationary point, that is $f(x_0) \neq x_0$ for every $x_0 \in S^1$. Consider the lift with $\tilde{f}(0) \in [0,1)$. Then also $\tilde{f}(x) - x \notin \mathbb{Z}$ for every $x \in [0,1)$, and by periodicity we conclude that $\tilde{f}(x_0) - x_0 \notin \mathbb{Z}$

for every $x \in \mathbb{R}$. By continuity, the assumption that $0 < \tilde{f}(0) < 1$ and compactness of the interval $[0, 1]$ this implies that there is a positive $\delta > 0$ such that

$$\delta \leq \tilde{f}(x_0) - x_0 \leq 1 - \delta$$

for every $x_0 \in \mathbb{R}$. Choosing $x_0 = \tilde{f}^n(0)$, $n \in \mathbb{N}$, we obtain

$$\delta \leq \tilde{f}^{n+1}(x_0) - \tilde{f}^n(x_0) \leq 1 - \delta.$$

for $n \in \mathbb{N}_0$. Summing over the first n, we obtain

$$n\delta \leq \tilde{f}^n(0) - 0 \leq n(1 - \delta)$$

for every $n \in \mathbb{N}$, and hence $\rho(f) \in [\delta, 1 - \delta]$. In particular, $\rho(f) \neq 0$. This concludes the proof. □

Later, we will consider diffeomorphisms of the form $f(x) = x + \rho + \eta(x)$, where η is periodic.

Lemma 2.13. *Let η be a continuous, periodic function. If the diffeomorphism $f(x) = x + \rho + \eta(x)$ satisfies $\rho(f) = \rho$, then there exists $x_0 \in S^1$ such that $\eta(x_0) = 0$.*

Proof. Observe by induction that

$$\tilde{f}^n(x_0) = x_0 + n\rho + \sum_{k=0}^{n-1} \eta \circ \tilde{f}^k(x_0).$$

Hence, if $\rho(f) = \rho$, we get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \eta \circ \tilde{f}^k(x_0) = 0.$$

But this must mean that η has a root since otherwise by continuity it would be strictly bounded away from zero. □

Exercise. Consider the diffeomorphism

$$f(x) = x + \frac{1}{2} + \frac{1}{4\pi} \sin(2\pi x).$$

(a) Compute $\rho(f)$.

(b) Find two periodic orbits.

(c) Find a non-periodic orbit.

2.1 Irrational rotation numbers and Denjoy's theorem

To understand the dynamic even better, we consider the set of all limit points of an orbit, the so-called ω-limit set.

Definition 2.14. Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and $x_0 \in S^1$. We define the ω*-limit set* of the orbit γ_{x_0} by

$$\omega(x_0) = \{y \in S^1 : \exists n_1 < n_2 < \dots \text{ s.t. } f^{n_k}(x_0) \xrightarrow[k\to\infty]{} y\}.$$

We can rephrase the result from Proposition Proposition 2.6.

Example 2.15. If $\alpha \in [0,1) \setminus \mathbb{Q}$ and $f = R_\alpha$ is the rotation, then

$$\omega(x_0) = S^1 \quad \text{for all } x_0 \in S^1.$$

That all orbits have the same limit set is no coincidence as the following theorem demonstrates.

Theorem 2.16. *Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and $\rho(f) \in [0,1) \setminus \mathbb{Q}$. Then $\omega(x_0) = \omega(y_0)$ for every $x_0, y_0 \in S^1$.*

Proof. Let $x_0 \in S^1$ and $x \in \omega(x_0)$. Let $\varepsilon > 0$ fixed. Then there must be a pair of integers $m > n$ such that

$$|f^m(x_0) - x| < \varepsilon \quad \text{and} \quad |f^n(x_0) - x| < \varepsilon.$$

Consider the closed interval $I = [f^m(x_0), f^n(x_0)] \subset [x-\varepsilon, x+\varepsilon]$ in S^1.
Claim: for every $y_0 \in S^1$, there exists $k \in \mathbb{N}$ such that $f^k(y_0) \in I$. Indeed, once we have demonstrated this claim, the proof of the theorem follows: since ε was chosen arbitrarily, this shows that we may construct a sequence $y_\ell = f^{k_\ell}(y_{\ell-1})$ starting from y_0 and such that $|y_\ell - x| < \frac{1}{\ell}$. From this we conclude that $x \in \omega(y_0)$ and since the assertion is symmetric in x_0 and y_0 the theorem.

It remains to prove the claim. We trace back, from where we can end up in I: define the sets $I_0 = I$ and

$$I_k = f^{-k(m-n)}(I) = [f^{kn-(k-1)m}(x), f^{(k+1)n-km}(x)],$$

hence the right endpoint of I_k is the left endpoint of I_{k+1} and the intervals wrap around the circle. Consider now the union of all of these intervals

$$\bigcup_{k=0}^{n} I_k.$$

This is one large, closed interval. We now want to show that there is N large enough so that $\bigcup_{k=0}^{n} I_k = S^1$. Suppose for the contrary, that this is not the case. Then the right endpoint of the interval must be bounded to the left of $f^m(x_0)$ and so

$$\lim_{k\to\infty} f^{-k(m-n)}(x_0) = p \in S^1$$

must exist. However, this cannot be since

$$\begin{aligned} p &= \lim_{k\to\infty} f^{-k(m-n)}(x_0) \\ &= \lim_{k\to\infty} f^{-(k-1)(m-n)}(x_0) \\ &= f^{(m-n)}\big(\lim_{k\to\infty} f^{-k(m-n)}(x_0)\big) \\ &= f^{(m-n)}(p), \end{aligned}$$

so that$p$ were a periodic point. But since the rotation number ρ was irrational, there are no periodic points. So, we conclude

$$S_1 = \bigcup_{k=0}^{N} I_k$$

for some $N \in \mathbb{N}$. For $y_0 \in S^1$ fixed, there must be $k \in \mathbb{N}$ with $y_0 \in I_k$ and so

$$f^{k(m-n)}(y_0) \in I.$$

This proves the claim and so the theorem. □

Next, we want to understand the structure of $\omega = \omega(x_0)$ for irrational rotation numbers $\rho(f) \notin \mathbb{Q}$.

Recall that a set C is called a *Cantor set* if it is a compact, totally disconnected set without isolated points.

Theorem 2.17. *Given any Cantor set $C \subset [0,1)$ and any $\rho \in [0,1) \setminus \mathbb{Q}$, there is an orientation-preserving homeomorphism $f \colon S^1 \to S^1$ such that*

$$\omega(x_0) = C \quad \textit{and} \quad \rho(f) = \rho.$$

Remark 2.18. One can show that for any orientation-preserving homeomorphism of the circle either $\omega = S^1$ or ω is a Cantor set.

We now introduce an equivalence class of dynamical systems.

Definition 2.19. Let $f, g \colon S^1 \to S^1$ be orientation-preserving diffeomorphisms. Then f and g are called *(topologically) conjugate* if there is a homeomorphism $h \colon S^1 \to S^1$ such that

$$g \circ h = h \circ f.$$

Note that we could have equivalently written that $f = h^{-1} \circ g \circ h$. If f and g are topologically conjugate, then all topological dynamical properties are the same. In particular, if $\omega \neq S^1$ and $\rho(f) \notin \mathbb{Q}$, then f cannot be conjugate to a rotation.

What are sufficient conditions for an orientation-preserving homeomorphism with $\rho(f) \notin \mathbb{Q}$ to be conjugate to the rotation R_ρ

We will cite here two theorems due to Denjoy: the first gives a positive answer under sufficient regularity, the second a negative answer under lack of said regularity.

Theorem 2.20 (Denjoy, 1932)**.** *If $f \colon S^1 \to S^1$ is a C^2-diffeomorphism with $\rho = \rho(f) \notin \mathbb{Q}$ irrational, then f is topologically conjugate to the rotation R_ρ. It suffices to assume that f' has bounded variation.*

Theorem 2.21 (Denjoy, 1946)**.** *Let $\rho \in [0,1) \setminus \mathbb{Q}$ and $\varepsilon > 0$. There exists a $C^{2-\varepsilon}$-diffeomorphism $f \colon S^1 \to S^1$ with $\rho(f) = f$ which is not conjugate to a rotation.*

With these results, we conclude the discussion about general diffeomorphisms on the circle.

3 KAM theory for analytic perturbations of rotations

The previous results show that C^2-diffeormophisms on the circle with irrational rotation number are topologically conjugate to an irrational rotation. But then h is merely a homeomorphism.

Question: can we say more about the regularity of h? For example, can we say that h is as regular as f? This is a very difficult questions for which the methods developed by Poincaré and Denjoy do not work. It will turn out that if we can even say something, then we also encounter a loss of regularity.

We will consider only small analytic perturbations of the rotations, that is we will consider analytic diffeomorphism $f\colon S^1 \to S^1$ of the form

$$f(x) = x + \rho + \eta(x) \bmod \mathbb{Z},$$

with $\rho \notin \mathbb{Q}$ and η periodic and analytic. Note that we can assume that $\rho(f) = \rho$: if $f(x) = x + \alpha + \mu(x)$ satisfies $\rho(f) = \rho$, then we can consider

$$\tilde{f}(x) = x + \rho + (\alpha - \rho + \mu(x)) =: x + \rho + \eta(x)$$

instead. We want to measure the regularity and size of η, as we think of η to be small.

Definition 2.22. For fixed $\sigma > 0$, we define the strip

$$S_\theta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \sigma\}$$

and we define the set of analytic functions

$$B_\sigma = \big\{\eta \in C(\overline{S}_\sigma; \mathbb{C}) : \eta \text{ is analytic in } S_\sigma,\ \eta(z) = \eta(z+1) \text{ for all } z \in S_\sigma \\ \text{and } \|\eta\|_\sigma < \infty\big\}\,.$$

Here, we define

$$\|\eta\|_\sigma = \sup_{z \in S_\sigma} |\eta(z)|.$$

We will assume that for some $\sigma > 0$

$$\eta \in B_\sigma \quad \text{and} \quad \|\eta\|_\sigma < \varepsilon,$$

where $\varepsilon > 0$ will be specified later[6].

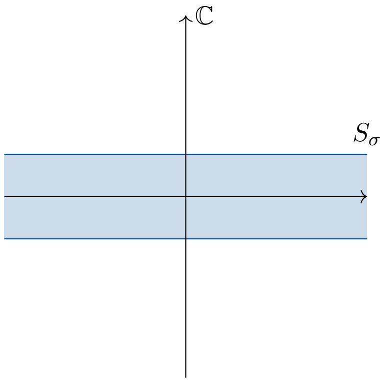

Fig. 2.1: The strip S_σ

[6] We will see that the change of coordinates h will only be analytic in $B_{\sigma-\delta}$ for some $\delta > 0$.

3.1 Heuristics and Diophantine numbers

We start by discussing a heuristical approach. The results of this will be used later to set up an iterative scheme.

Recall that we consider the diffeomorphism

$$f(x) = x + \rho + \eta(x).$$

We look for an analytic function $h\colon S^1 \to S^1$ such that

$$f \circ h = h \circ R_\rho.$$

Making the ansatz that

$$h(x) = x + H(x),$$

where H is an analytic and periodic function, we find that

$$f(x + H(x)) = x + H(x) + \rho + \eta(x + H(x)) = x + \rho + H(x + \rho) = h \circ R_\rho(x).$$

Rearranging this, we obtain

$$H(x + \rho) - H(x) = \eta(x + H(x)) = \eta(x) + \eta'(x)H(x) + \text{h.o.t.} \approx \eta(x).$$

Consequently, as an approximation, we first consider the equation

$$H(x + \rho) - H(x) = \eta(x). \tag{2.3}$$

Then, since H and η are both periodic, we can apply the Fourier transform[7] on both sides to obtain

$$e^{2\pi i n\rho}\hat{H}_n - \hat{H}_n = \hat{\eta}_n, \quad n \in \mathbb{Z}.$$

[7] $\hat{\eta}_n = \int_0^1 \eta(x)e^{-2\pi i n x}\,\mathrm{d}x$

Since $\rho \notin \mathbb{Q}$, it is $e^{2\pi i n\rho} - 1 \neq 0$ for every $n \neq 0$, hence we obtain

$$H(x) = \sum_{n \neq 0} \frac{\hat{\eta}_n}{e^{2\pi i n\rho} - 1} e^{2\pi i n x}. \tag{2.4}$$

Two problems ensue from this definition:

1) Equation (2.3) does not hold since

$$H(x + \rho) - H(x) = \sum_{n \neq 0} \hat{\eta}_n e^{2\pi i n x} = \eta(x) - \hat{\eta}_0 = \eta(x) - \int_0^1 \eta(y)\,\mathrm{d}y. \tag{2.5}$$

   We can deal with this problem later in the proof.

2) Does the series in (2.4) even have a chance to converge? The problem is that $e^{2\pi i n\rho} - 1$ might be very small very often.

The second problem is called the *problem of small denominators*. This will appear again when we study the n-body problem of celestial mechanics. For a given rotation number $\rho \notin \mathbb{Q}$, we don't know how well-behaved these denominators are. But, we can resort to number theory again.

Last time, we have seen that any $\rho \in \mathbb{R}$ could be approximated by rational numbers such that

$$\left|\rho - \frac{p}{q}\right| < \frac{1}{q^2}.$$

It turns out that some numbers can be approximate better than others. This leads us to the Diophantine classes of numbers.

Definition 2.23. The irrational number ρ is of *Diophantine type* (K, ν) with $K, \nu > 0$ if

$$\left|\rho - \frac{p}{q}\right| > K|q|^{-\nu}$$

holds true for all $(p, q) \in \mathbb{Z} \times \mathbb{N}$.

We can w.l.o.g. assume that $K \leq 1$. In particular, there is no number of Diophantine type $(1, 2)$, but generically that is the best we can do approximating a number as the following proposition demonstrates.

Proposition 2.24. *For every $\nu > 2$, almost every irrational number ρ is of type (K, ν) for some $K > 0$.*

Proof. Let $\nu > 2$. It is enough to show that every irrational number $\rho \in [0, 1]$ is of the type (K, ν) for some $K > 0$.

First fix $K > 0$ and $\frac{p}{q} \in \mathbb{Q}$. Then define

$$I_{K,p,q} := \{\rho \in [0, 1] \setminus \mathbb{Q} : |\rho - \frac{p}{q}| \leq K|q|^{-\nu}\}.$$

We will show that

$$\left| \bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{N}} \bigcup_{p=1}^{q} I_{\frac{1}{n},p,q} \right| = 0. \tag{2.6}$$

First note that $I_{K,p,q}$ is essentially an interval up to a set of measure zero, hence we obtain

$$|I_{K,p,q}| \leq 2K|q|^{-\nu}.$$

But this implies, taking the union over all possible $p = 1, \ldots, q$ that still for fixed q

$$\left| \bigcup_{p=1}^{q} I_{K,p,q} \right| \leq 2K|q|^{-\nu+1}.$$

Now, taking the union over all possible q, we find

$$\left| \bigcup_{q\in\mathbb{N}} \bigcup_{p=1}^{q} I_{K,p,q} \right| \leq 2K \sum_{q\in\mathbb{N}} q^{-\nu+1} < 2cK$$

for some $c > 0$ since $-\nu + 1 < 1$ and hence we obtain convergence of the series. But this readily implies (2.6). □

We can now use the Diophantine type of a number to get explicit control of the small denominator.

Lemma 2.25. *If $\rho \in [0, 1) \setminus \mathbb{Q}$ is of Diophantine type (K, ν), then*

$$|e^{2\pi in\rho} - 1| \geq 4K|n|^{-(\nu-1)}$$

for all $n \neq 0$.

Proof. Let $m \in \mathbb{N}$. Then using $e^{2\pi im} = 1$, we obtain

$$\begin{aligned} |e^{2\pi in\rho} - 1| &= |e^{2\pi im}(e^{2\pi i(n\rho-m)} - 1)| \\ &= |e^{2\pi i(n\rho-m)} - 1| \\ &= 2|\sin(\pi(\rho n - m))|, \end{aligned}$$

where we used in the last step that $|e^{ix} - 1| = 2\sin(x/2)$. Using the inequality

$$|\sin(\pi x)| \geq 2|x|, \quad |x| \leq \frac{1}{2},$$

we conclude that

$$|e^{2\pi i n\rho} - 1| \geq 4|\rho n - m| \geq 4K|n|^{-(\nu-1)}.$$

This concludes the proof. □

Using Cauchy's theorem, we get exponential decay for the Fourier coefficients of an analytic function.

Lemma 2.26. *Let $\eta \in B_\sigma$. Then*

$$|\hat{\eta}_n| \ \leq \|\eta\|_\sigma e^{-2\pi\sigma|n|}. \tag{2.7}$$

Proof. Recall that

$$\hat{\eta}_n = \int_0^1 \eta(x) e^{-2\pi i n x}\,\mathrm{d}x.$$

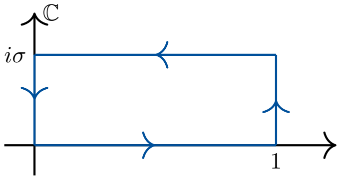

Fig. 2.2: The contour C

Denote by C the contour in $\mathbb{C}$ given by concatenating the path $[0,1]$, $[1, 1 \pm i\sigma]$, $[1 \pm i\sigma, \pm i\sigma]$ and $[\pm i\sigma, 0]$. By Cauchy's integral theorem, it holds

$$\int_C \eta(z) e^{-2\pi i n z}\,\mathrm{d}z = 0.$$

Combining this with periodicity, we find that

$$\int_{[0,1]} \eta(z) e^{-2\pi i n z}\,\mathrm{d}z = \int_{[\pm i\sigma, 1 \pm i\sigma]} \eta(z) e^{-2\pi i n z}\,\mathrm{d}z = \int_0^1 \eta(x \pm i\sigma) e^{2\pi i n x} e^{\mp 2\pi n\sigma}\,\mathrm{d}x.$$

Choosing either the pass on the upper or lower halfplane, depending on the sign of n, we obtain

$$|\hat{\eta}_n| \leq \|\eta\|_\sigma e^{2\pi|n|\sigma}$$

which concludes the proof. □

Remark 2.27. Also the inverse statement is true. Assume that (2.7) holds true, then the function

$$\eta(x) = \sum_{n\in\mathbb{Z}} \hat{\eta}_n e^{2\pi i n x}$$

is analytic in the strip S_σ and

$$\|\eta\|_{\sigma-\delta} \leq \frac{C}{\delta}.$$

We may apply these results to the function

$$H(x) = \sum_{n\neq 0} \frac{\hat{\eta}_n}{e^{2\pi i n\rho} - 1} e^{2\pi i n x}.$$

and obtain the following bound.

Proposition 2.28. *If ρ is of type (K,ν) and $\eta \in B_\sigma$ and $\delta < \sigma$, then $H \in B_{\sigma-\delta}$ with*

$$\|H\|_{\sigma-\delta} \le \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma, \tag{2.8}$$

where Γ denotes the standard Γ-function

$$\Gamma(\nu) = \int_0^1 x^{\nu-1}e^{-x}\,\mathrm{d}x.$$

Proof. It suffices to prove absolute convergence of the series locally uniformly in $S_{\sigma-\delta}$, then analyticity follows. Indeed, using Lemmas 2.25 and 2.26, we obtain for $z \in S_{\sigma-\delta}$

$$\begin{aligned}
|H(z)| &\le \sum_{n\neq 0} \frac{|\hat{\eta}_n|}{|e^{2\pi i n\rho}-1|}|e^{2\pi i n z}| \\
&\le \sum_{n\neq 0} \frac{|n|^{\nu-1}}{4K}\|\eta\|_\sigma e^{-2\pi\sigma\ |n|}e^{2\pi|n|(\sigma-\delta)} \\
&\le \frac{\|\eta\|_\sigma}{4K}\sum_{n\neq 0}|n|^{\nu-1}e^{-2\pi\delta|n|} \\
&= \frac{\|\eta\|_\sigma}{2K}\sum_{n=1}^{\infty} n^{\nu-1}e^{-2\pi\delta n} \\
&\le \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma.
\end{aligned}$$

Here, in the last step, we use

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\nu-1}e^{-2\pi\delta n} &\le \int_0^\infty y^{\nu-1}e^{-2\pi\delta y}\,\mathrm{d}y \\
&= \frac{1}{(2\pi\delta)^\nu}\int_0^\infty x^{\nu-1}e^{-x}\,\mathrm{d}x \\
&= \frac{1}{(2\pi\delta)^\nu}\Gamma(\nu),
\end{aligned}$$

which completes the estimate. □

Next, we need to make sure that we have actually constructed an analytic diffeomorphism.

Proposition 2.29. *Assume that $2\delta < \sigma$ and*

$$\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1.$$

The map $h(z) = z + H(z)$ is analytic with analytic inverse on the domain $S_{\sigma-2\delta}$. Furthermore, h^{-1} is well-defined on $S_{\sigma-3\delta}$.

To prove the proposition, we rely on another lemma from complex analysis.

Lemma 2.30. *Let $\eta \in B_\sigma$. Then for every $0 < \delta < \sigma$ it holds*

$$\|\eta'\|_{\sigma-\delta} \le \frac{2\pi}{\delta}\|\eta\|_\sigma.$$

Proof. From Cauchy's integral formula, we know that for every $z \in S_{\sigma-\delta}$ and every ball $B_r(z) \subset S_\sigma$, we may write

$$\eta'(z) = \int_{\partial B_r(z)} \frac{\eta(w)}{(w-z)^2} \, \mathrm{d}w.$$

Hence, we obtain the estimate

$$|\eta'(z)| \leq \|\eta\|_\sigma \int_{\partial B_r(z)} \frac{dw}{r^2} = \frac{2\pi}{r} \|\eta\|_\sigma.$$

Sending r to δ proves the claim. □

Proof of Proposition 2.29. Analyticity of H and thus of h on $S_{\sigma-\delta}$ follows by construction. We only need to make sure that h is invertible. Recall that it is enough to find a domain so that $\|H'\| < 1$ holds true as this guarantees injectivity.

Now Lemma 2.30 and Proposition 2.28 imply that

$$\|H'\|_{\sigma-2\delta} \leq \frac{2\pi}{\delta} \|H\|_{\sigma-\delta} \leq \frac{(2\pi)^2 \Gamma(\nu)}{K(2\pi\delta)^{\nu+1}} \|\eta\|_\sigma < 1. \tag{2.9}$$

Hence, h restricted to $S_{\sigma-2\delta}$ is a diffeomorphism onto its image.

For the second part of the claim, we realise that by Proposition 2.28 and the assumption

$$\|H\|_{\sigma-\delta} \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu} \|\eta\|_\sigma < \delta. \tag{2.10}$$

□

To summarise: we have found an analytic change of coordinates $h(x) = x + H(x)$. Two problems remain though:

1) h is not the correct change of variables, as we have ignored higher order terms. Intuitively though, h is a step into the right direction and we may try to iterate the procedure.

2) If we constantly lose a fixed δ in analyticity during the iteration, we cannot have hope of constructing an analytic diffeomorphism. We need to choose a clever iteration scheme.

3.2 Main theorem and Newton's method

Before it is time to discuss how we can iterate the procedure introduced above, it is time to state the main theorem.

Theorem 2.31 (Arnold 1961)**.** *Assume that ρ is of Diophantine type (K, ν) and $\sigma > 0$. Then there exists $\varepsilon = \varepsilon(K, \nu, \sigma) > 0$ such that if*

$$f(x) = x + \rho + \eta(x)$$

has rotation number ρ and $\eta \in B_\sigma$ satisfies $\|\eta\|_\sigma < \varepsilon$, then there exists an analytic and invertible change of variables h which conjugates f to the rotation R_ρ:

$$R_\rho = h^{-1} \circ f \circ h.$$

Recall that so far we have heuristically taken the following approach: we want to find $h = \mathrm{Id} + H$ such that

$$H \circ R_\rho - R_\rho = \eta \circ h \approx \eta,$$

that is, we have linearised the equation to find a solution. Compare this to Newton's method:

Consider a map $F \in C^2(\mathbb{R})$ and we look for a root $\bar{x}$ of F. To do so, consider some point x_0 and define the linearisation around x_0 by

$$L_{x_0}(x) = F(x_0) + F'(x_0)(x - x_0).$$

We can find a root of L_{x_0} (provided $F'(x_0) \neq 0$), that is

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}.$$

Now, we want to iterate this and set

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.$$

For this to work, we require that $F' \neq 0$ in a neighborhood of the root $\bar{x}$. We can show very fast convergence: applying Taylor's theorem, we get

$$L_y(\bar{x}) - F(\bar{x}) = \frac{1}{2} F''(\xi)(\bar{x} - y)^2$$

for some $\xi \in [\bar{x}, y]$. Now set $\varepsilon_n = |\bar{x} - x_n|$. From the formula, we conclude that there is $\xi_n \in [\bar{x}, x_n]$ such that

$$F(x_n) - (\bar{x} - x_n)F'(x_n) = \frac{1}{2} F''(\xi_n)(x_n - \bar{x})^2.$$

Dividing by $F'(x_n)$ and using the definition of x_{n+1}, this leads to

$$x_{n+1} - \bar{x} = \frac{F''(\xi_n)}{2F'(x_n)}(x_n - \bar{x})^2$$

If $|F'|$ is bounded from below in a neighborhood of $\bar{x}$ and since F'' is bounded from above, there is a constant $C > 0$ such that

$$\varepsilon_{n+1} \leq C\varepsilon_n^2$$

and hence by iteration

$$|x_n - \bar{x}| \leq C\varepsilon_0^{2n}$$

superexponential convergence of x_n to $\bar{x}$.

Now, with Newton's method in mind, we will prove Theorem 2.31 by an iterative argument relying on the heuristics studied in the previous subsection. In order to achieve this, we need to understand $f_1 = h^{-1} \circ f \circ h$, where $h(x) = x + H(x)$ is the analytic change of variables constructed before. This is part of the following two propositions. They make rigorous our intuitive guess that $f_1 = R_\rho + \tilde{\eta}$, where $\|\tilde{\eta}\| \lesssim \|\eta\|_\sigma^2$.

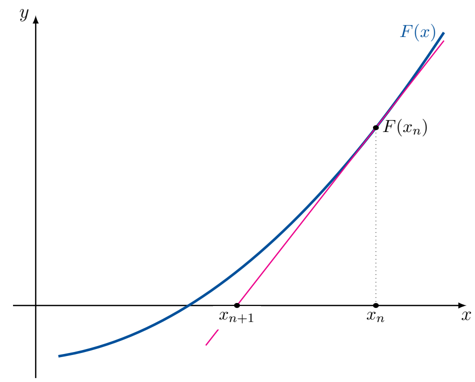

Fig. 2.3: Iteration step for Newton's method

Proposition 2.32. *Assume that* $4\delta < \sigma$ *and*

$$\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1.$$

Then it holds $h^{-1}(z) = z - H(z) + g(z)$, *where*

$$\|g\|_{\sigma-4\delta} \leq \frac{(2\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2.$$

Proof. Define g by

$$g(z) = h^{-1}(z) - z + H(z).$$

Then using that $h^{-1} = z - H(z) + g(z)$, we obtain for every $z \in S_{\sigma-2\delta}$

$$z = h^{-1} \circ h(z) = h^{-1}(z + H(z)) = z + H(z) - H(z + H(z)) + g(z + H(z)). \quad (2.11)$$

Solving for g, we find

$$g(z + H(z)) = H(z + H(z)) - H(z) = \int_0^1 H'(z + sH(z))H(z)\,\mathrm{d}s.$$

So, we may define g for each $\xi = h(z)$ by

$$g(\xi) = H(h^{-1}(\xi))\int_0^1 H'(h^{-1}(\xi) + sH(h^{-1}(\xi)))\,\mathrm{d}s.)$$

As in the proof of Proposition 2.29, we may argue using $\|h\|_{\sigma-\delta} < \delta$ that since the image of $S_{\sigma-3\delta}$ under h must contain $S_{\sigma-4\delta}$, we have for $\xi \in S_{\sigma-4\delta}$ that $h^{-1}(\xi) \in S_{\sigma-3\delta}$ and furthermore $h^{-1}(\xi) + sH(h^{-1}(\xi)) \in S_{\sigma-2\delta}$ so that we may apply the estimates from equations (2.8) and (2.9) to obtain

$$\|g\|_{\sigma-4\delta} \leq \|H\|_{\sigma-\delta}\|H'\|_{\sigma-2\delta} \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma.$$

This proves the claim. □

Finally, we need to understand how large is the error that we make by linearisation. Therefore, define $f_1 = h^{-1} \circ f \circ h$. Intuitively, f_1 is closer to a rotation and we expect from Newton's method that the error is also quadratic in terms of $\|\eta\|_\sigma^2$. This is demonstrated in the following proposition.

Proposition 2.33. *Assume that* $4\delta < \sigma$ *and*

$$\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1.$$

Then $f_1(x) = h^{-1} \circ f \circ h(x) = x + \rho + \eta_1(x)$, *where*

$$\|\eta_1\|_{\sigma-6\delta} \leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2.$$

Proof. Using the previous results, we can spell out f_1 explicitly:

$$\begin{aligned} f_1(x) &= h^{-1} \circ f \circ h(x) \\ &= h^{-1}(x + H(x) + \rho + \eta(x + H(x))) \\ &= x + H(x) + \rho + \eta(x + H(x)) - H(x + H(x) + \rho + \eta(x + H(x))) \\ &\quad + g(x + H(x) + \rho + \eta(x + H(x))) \\ &= x + \rho + [H(x) - H(x + \rho) + \eta(x)] + [\eta(x + H(x)) - \eta(x)] \\ &\quad + [H(x + \rho) - H(x + \rho + H(x) + \eta(x + H(x))] \\ &\quad + g(x + H(x) + \rho + \eta(x + H(x))). \end{aligned} \tag{2.12}$$

Now define

$$\eta_1(x) = f_1(x) - x - \rho.$$

We want to obtain an estimate for η_1. The reason for splitting the term in this particular fashion becomes clear immediately: the first bracket was the linearised equation for H, see (2.5), so we find

$$H(x) - H(x + \rho) + \eta(x) = \hat{\eta}_0.$$

The second bracket we may rewrite as

$$\eta(x + H(x)) - \eta(x) = \int_0^1 \eta'(x + sH(x))H(x)\,\mathrm{d}s$$

and as before we obtain a bound for this term using the assumptions

$$\|\eta(z + H(z)) - \eta(z)\|_{S_{\sigma-4\delta}} \leq \|H\|_{\sigma-\delta}\|\eta'\|\sigma - 2\delta \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma \frac{2\pi}{\delta}\|\eta\|_\sigma. \tag{2.13}$$

The same applies to the third bracket:

$$\begin{aligned} &H(z + \rho + H(z) + \eta(z + H(z)) - H(z + \rho) \\ &= \int_0^1 H'(z + \rho + s(H(z) + \eta(z + H(z))))(H(z) + \eta(z + H(z)))\,\mathrm{d}s. \end{aligned}$$

We can bound the norm of this term for $z \in S_{\sigma-4\delta}$ using the assumed bounds:

$$\|H(z + \rho + H(z) + \eta(z + H(z)) - H(z + \rho)\|_{\sigma-4\delta} \leq (\|H\|_{\sigma-\delta} + \|\eta\|_\sigma)\|H'\|_{\sigma-2\delta}.$$

We obtain the explicit bound

$$\begin{aligned} (\|H\|_{\sigma-\delta} + \|\eta\|_\sigma)\|H'\|_{\sigma-2\delta} &\leq \left(\frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma + \|\eta\|_\sigma\right)\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma \\ &\leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2. \end{aligned} \tag{2.14}$$

For the last term, notice that if $z \in S_{\sigma-6\delta}$, then $z + H(z) + \rho + \eta(z + H(z)) \in S_{\sigma-4\delta}$ and we may apply Proposition 2.32 to obtain

$$\|g(z + H(z) + \rho + \eta(z + H(z)))\|_{\sigma-6\delta} \leq \frac{(2\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2. \tag{2.15}$$

We are left with finding a corresponding bound for $\hat{\eta}_0$. Since f_1 and f are conjugate, the rotation numbers must be equal $\rho(f_1) = \rho(f) = \rho$. We may apply Lemma 2.13 to f_1 and find that there must be $x_0 \in [0, 1)$ such that $\eta_1(x_0) = 0$ and hence $f_1(x_0) = x_0 + \rho$. Using this expression in (2.12), we obtain

$$\begin{aligned}\hat{\eta}_0 = -[\eta(x + H(x)) - \eta(x)]& \\ &- [H(x + \rho) - H(x + \rho + H(x) + \eta(x + H(x))] \\ &- g(x + H(x) + \rho + \eta(x + H(x))).\end{aligned}$$

We have already bounded each of the objects on the right-hand side in (2.13), (2.14) and (2.15) so that we obtain

$$|\hat{\eta}_0| \leq \frac{(2\pi^2)\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma^2 + \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2 + \frac{(2\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2. \tag{2.16}$$

Combining the four estimates (2.13), (2.14), (2.15) and (2.16), the theorem follows. □

Before we come to the technical details of the iteration scheme, this is a good time to summarise our approach.

1. Linearisation. We want to solve $h \circ R_\rho = f \circ h$, that is

$$H \circ R_\rho - H = \eta \circ h,$$

where we write $h = \mathrm{Id} + H$. Then we linearise this equation and obtain the approximate equation

$$H \circ R_\rho - H = \eta.$$

2. Solving the linear equation. We solved the linear equation for analytic right-hand side η and obtained corresponding bounds for H in a smaller strip in Proposition 2.28. In Proposition 2.29 we showed that we obtained a diffeomorphism.

3. Error term in linearised solution. Defining $f_1 = h^{-1} \circ f \circ h$, we miss R_ρ by an error which is of the order $\|\eta\|_\sigma^2$. In the process, we lose regularity.

4. Fast convergence due to Newton scheme. If we define inductively $f_{n+1} = h_n^{-1} \circ f_n \circ h_n$, then the fast convergence of the Newton scheme will allow us to show that the maps

$$h_0 \circ h_1 \circ \cdots \circ h_n$$

actually converge to an analytic diffeomorphism of the circle conjugating f to the rotation R_ρ.

Now we are in the position to iterate this scheme. We define

$$f_0(x) = f(x) = R_\rho(x) + \eta(x), \quad \eta_0(x) = \eta(x). \tag{2.17}$$

We define $h_0(x) = h(x) = x + H(x)$ and

$$f_1(x) = h_0^{-1} \circ f_0 \circ h_0 = R_\rho + \eta_1.$$

Then we inductively define

$$f_{n+1} = h_n^{-1} \circ f_n \circ h_n = R_\rho + \eta_{n+1},$$

where $h_n = \mathrm{Id} + H_n$ are constructed as before, i.e. they solve

$$H_n(x+\rho) - H_n(x) = \eta_n(x) - \hat{\eta}_n(0).$$

Define for $\sigma_0 = \sigma$ and $\varepsilon_0 = \|\eta\|_\sigma$ the inductive constants:

$$\begin{aligned} \delta_n &= \frac{\sigma}{36(1+n^2)}, \\ \sigma_{n+1} &= \sigma_n - 6\delta_n, \\ \varepsilon_n &= \varepsilon_0^{(3/2)^n} \end{aligned}$$

if $n \geq 0$. We also define

$$\sigma^* = \lim_{n\to\infty} \sigma_n > \frac{\sigma}{2} > 0.$$

We need to make sure that these constants are chosen so that our inductive scheme works correctly before we may prove the main theorem.

Lemma 2.34. *If*

$$\|\eta\|_\sigma = \varepsilon_0 < \left(\frac{K}{16\pi\Gamma(\nu)} \left(\frac{\sigma}{36}\right)^{\nu+1} \right)^8,$$

then $f_{n+1}(x) = x + \rho + \eta_{n+1}(x)$ *with* $\eta_{n+1} \in B_{\sigma_{n+1}}$ *and*

$$\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \varepsilon_{n+1}.$$

Furthermore, $h_n = \mathrm{Id} + H_n$ *satisfies*

$$\|H_n\|_{\sigma_n - \delta_n} \leq \frac{\Gamma(\nu)\varepsilon_n}{K(2\pi\delta_n)^\nu},$$

and $h_n^{-1} = x - h_n + g_n$*, where*

$$\|g_n\|_{\sigma_n - 4\delta_n} \leq \frac{(2\pi)^2\Gamma(\nu)^2\varepsilon_n^2}{K^2(2\pi\delta_n)^{(2\nu+1)}}.$$

Proof. For $n = 0$, the estimates for H_n and g_n were demonstrated in Propositions 2.28 and 2.32. Proposition 2.33 gives the estimate

$$\begin{aligned} \|\eta_1\|_{\sigma-6\delta} &\leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}} \|\eta\|_\sigma^2 < \varepsilon_0^{3/2} \left(\frac{K}{16\pi\Gamma(\nu)} \left(\frac{\sigma}{36}\right)^{\nu+1} \right)^4 \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}} \\ &\leq \varepsilon_0^{3/2}. \end{aligned}$$

Now suppose the induction holds up to step $n-1$ so that we know that $\|\eta_n\|_{\sigma_n} \leq \varepsilon_n$. Then Propositions 2.28 and 2.32 give the corresponding bounds for H_n and g_n, respectively. Again by Proposition 2.33, we find

$$\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta_n)^{(2\nu+1)}} \varepsilon_n^2 \leq \varepsilon_{n+1}^{3/2}$$

as before. □

Now we can finally prove Arnold's theorem.

Proof of Theorem 2.31. Define the change of coordinates via $\psi_0 = h_0$ and

$$\psi_n = h_n \circ \psi_{n-1} = h_n \circ \cdots \circ h_1 \circ h_0.$$

Then it is

$$\begin{aligned}\psi_n(x) &= x + H_n(x) + h_{n-1}(x + H_n(x)) + h_{n-2}(x + H_n(x) + h_{n-1}(x + H_n(x))) \\ &\quad + \cdots + h_0(x + H_1(x + \cdots) + \cdots).\end{aligned}$$

From Lemma 2.34, we know that ψ_n is analytic on $S_{\sigma_n - 2\delta_n}$ and also

$$\|\psi_n - \operatorname{Id}\|_{\sigma_n - 2\delta_n} \le \sum_{k=0}^{\infty} \frac{\Gamma(\nu)\varepsilon_k}{K(2\pi\delta_n)^\nu} =: \Delta < \infty.$$

We need to show that ψ_n converges to an analytic limit. For this, note that

$$\begin{aligned}\psi_{n+1}(z) - \psi_n(z) &= \psi_n \circ h_{n+1}(z) - \psi_n(z) = \psi_n(z + H_{n+1}(z)) - \psi_n(z) \\ &= \int_0^1 \psi_n'(z + sH_{n+1}(z))H_{n+1}(z)\,\mathrm{d}s.\end{aligned}$$

We may use that $\psi_n - \operatorname{Id}$ is bounded, to obtain the bound $\|\psi_n'\|_{\sigma_n - 4\delta_n} \le \|(\psi_n - \operatorname{Id})'\|_{\sigma_n - 4\delta_n} + 1 \le \frac{2\pi\Delta}{\delta_n} + 1$ and hence

$$\|\psi_{n+1} - \psi_n\|_{\sigma_{n+1}} \le \|H_{n+1}\|_{\sigma_{n+1}}\|\psi_n'\|_{\sigma_{n+1}} \le \big(\frac{2\pi\Delta}{\delta_n} + 1\big)\frac{\Gamma(\nu)\varepsilon_{n+1}}{K(2\pi\delta_{n+1})^\nu}.$$

The series

$$\sum_{n=0}^{\infty}\big(\frac{2\pi\Delta}{\delta_n} + 1\big)\frac{\Gamma(\nu)\varepsilon_{n+1}}{K(2\pi\delta_{n+1})^\nu} \tag{2.18}$$

converges since ε_n is converging exponentially fast. Hence, the sequence $(\psi_n)_n$ is Cauchy in B_{σ^*} and converges there uniformly to a limit $h \in B_\sigma$[8]. We may write

[8] Note that uniform limits of analytic functions are analytic.

$$h(z) = z + H(z)$$

and find that

$$\|H'\|_{\sigma^* - \delta^*} \le \frac{\Delta}{\delta^*} < \delta^* \tag{2.19}$$

provided $\delta^* < \min\{\frac{\sigma^*}{16}, 1\}$. But this implies, since $\delta^* < 1$ that h is invertible on the image of $S_{\sigma^* - \delta^*}$ and that this image contains $S_{\sigma^* - 2\delta^*}$. We conclude the proof by noticing that by induction and using $f_0 = f$ together with $f_n \circ h_{n+1} = h_{n+1} \circ f_{n+1}$ it holds

$$f \circ \psi_n = \psi_n \circ f_n.$$

We may use this to obtain

$$f \circ h(z) = \lim_{n\to\infty} f \circ \psi_n(z) = \lim_{n\to\infty} \psi_n \circ f_n(z) = \lim_{n\to\infty} \psi_n(z + \rho + \eta_n(z)) = h \circ R_\rho(z)$$

due to the uniform convergence of $\psi_n \to h$ and η_n to zero. This proves that we have indeed constructed an analytic diffeomorphism h that conjugates f to the rotation R_ρ.

□

The restriction of the theorem to consider only diffeomorphism $f(x) = x + \rho + \eta(x)$ with rotation number equal to ρ seems, at first glance, to be a restriction. But this is not the case. In fact, consider a diffeomorphism

$$f(x) = x + \alpha + \mu(x)$$

for μ analytic with $\rho(f) = \rho$. Then, we can rearrange

$$f(x) = x + \rho + \big(\alpha - \rho + \mu(x)\big) \tag{2.20}$$

and apply Theorem 2.31 to the function $\eta(x) = \alpha - \rho + \mu(x)$[9]. This leads to the collowing corollary.

[9] Observe that we made explicit use of this specific form in the proof of Proposition 2.33, when estimating $|\hat{\eta}_0|$.

Corollary 2.35. *Assume that ρ is of Diophantine type (K, ν) and $\sigma > 0$. Then there exists $\varepsilon = \varepsilon(K, \nu, \sigma) > 0$ such that if*

$$f(x) = x + \alpha + \mu(x)$$

has rotation number ρ and $\mu \in B_\sigma$ satisfies $\|\alpha - \rho + \mu\|_\sigma < \varepsilon$, then f is conjugated to the rotation R_ρ by an analytic change of coordinates.

In practice, the computation of the rotation number is tedious and difficult work. We may overcome this problem by introducing another parameter into the system. For $\eta \in B_\sigma$ for some $\sigma >= 0$, $\varepsilon \in \mathbb{R}$ small and $\alpha \in [0, 1)$, define

$$f_{\alpha,\varepsilon}(x) = x + \alpha + \varepsilon\eta(x).$$

While, we do not know the rotation number, Arnold proved that a slightly changed diffeomorphism is still analytically conjugate to the rotation, cf. [Arn09, Theorem 2].

Theorem 2.36. *Assume that α is of Diophantine type (K, ν) and $\sigma > 0$. Assume that $\eta \in B_\sigma$ and let*

$$f_{\alpha,\varepsilon}(x) = x + \alpha + \varepsilon\eta(x).$$

Then there exists $\varepsilon_0 = \varepsilon_0(K, \nu, \sigma) > 0$ and an analytic function $\Delta(\varepsilon)$ such that

$$f_{\alpha,\varepsilon,\Delta(\varepsilon)}(x) = x + \alpha + \Delta(\varepsilon) + \varepsilon\eta(x).$$

is analytically conjugate to the rotation R_α for every $|\varepsilon| < \varepsilon_0$. The change of coordinates also depends analytically on ε.

Combing Theorem 2.36 with basic measure-theoretical considerations, we obtain the following result, cf. [Arn09, Theorem 8].

Theorem 2.37. *Let $\eta \in B_\sigma$ and for $\alpha \in [0, 1]$ and ε define*

$$f_{\alpha,\varepsilon,\Delta(\varepsilon)}(x) = x + \alpha + \Delta(\varepsilon) + \varepsilon\eta(x).$$

Then for every $\delta > 0$ there exists $\varepsilon_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$, there is a set $\mathcal{A}(\varepsilon) \subset [0, 1]$ such that

1) *for every* $\alpha \in \mathcal{A}(\varepsilon)$ *the diffeomorphism* $f_{\alpha,\varepsilon}$ *is analytically conjugated to a rotation,*

2) $|\mathcal{A}(\varepsilon)| > 1 - \delta$.

Finally, we remark on a counterexample due to Yoccoz, cf. [Wal95]. This shows that there are irrational rotation numbers for which we can find analytic diffeomorphisms of the circle that are not analytically conjugated to the rotation.

Theorem 2.38. *Let* $a > 3$. *There exists an irrational number* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *such that the analytic diffeomorphism*[10]

$$f_{a,\lambda}(z) = \lambda z^2 \frac{z+a}{1+az},$$

where $\lambda \in [0,1)$ *is the unique number such that* $\rho(f_{a,\lambda}) = \alpha$*, is not analytically conjugate to the rotation* R_α.

[10] f is called the Blaschke product.

This concludes the discussion of the KAM theory for analytic diffeomorphisms of the circle. We will use the local methods introduced here to discuss nearly-integrable Hamiltonian systems in Chapter 3.

3

KAM theory for nearly-integrable Hamiltonian systems

After having discussed the very specific case of diffeomorphisms on the circle and understood the local (analytic) picture of small perturbations of rotations, we turn to nearly-integrable Hamiltonian systems. We will see that for a class of so-called integrable Hamiltonian system, the motion of the particles is quasi-periodic on invariant tori. A nearly-integrable system is then a small-perturbation of an integrable system and similarly to the situation of the circle diffeomorphism, we may ask the question whether the perturbation changes the dynamics.

Before we can move to the KAM story for Hamiltonians, we begin by giving an introduction into Hamiltonian mechanics. Furthermore, we need to discuss the consequences of integrabilty and prove the Arnold–Liouville theorem transforming an integrable Hamiltonian system into a new set of variables, so-called action-angle variables, for which the dynamics is trivial.

1 Hamiltonian systems

1.1 Derivation, definition and first properties

Consider a a particle moving on a curve $q\colon [0, t_0] \to \mathbb{R}^3$ and define the state space $X = \mathbb{R}^3$. The motion of the particle is described by Newton's second law

$$F = ma,$$

where F denotes a force vector field $F\colon X \to X$, m denotes the mass of the particle and $a = \frac{d^2}{dt^2} q =: \ddot{q}$ denotes the acceleration. Assuming that forces are conservative[1], we may write $F(q) = -\nabla V(q)$ for a potential V. Defining the kinetic energy[2] by

$$K(\dot{q}) = \frac{m}{2} |\dot{q}|^2,$$

the *Lagrangian* is defined by

$$\mathcal{L}(q, \dot{q}) = K(\dot{q}) - V(q) = \frac{m}{2} \|\dot{q}\|^2 - V(q). \tag{3.1}$$

Hamiltonian's principle of least action now states that an system will always choose the path between two points that minimises the *action integral*

$$S = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}) \, \mathrm{d}t.$$

[1] That means the total work in moving the particle between two points is independent of the path.

[2] Note that $\frac{dK}{dt} = m\langle \dot{q}, \ddot{q} \rangle = \langle \dot{q}, F \rangle$ has a natural interpretation as the rate of work equation.

The Euler-Lagrange equations of S that hold when $\delta S = 0$ are given by

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot q} - \frac{\partial \mathcal{L}}{\partial q} = 0$$

and they are equivalent to Newton's second law.

We also verify that along solutions to the Euler-Lagrange equation, the energy is conserved, that is $dE/dt = 0$, where

$$E = \frac{m}{2}|\dot q|^2 + V(q).$$

This is known as the Lagrangian approach to mechanics. It has the advantage to be based on variation principles, but it requires to have a fixed starting and end point.

Note that we again only consider autonomous systems. In general, $\mathcal{L}$ can depend on time, but we will restrict the discussion on time-independent force fields.

To obtain Hamiltonian mechanics, we make the change of variables

$$p = \frac{\partial \mathcal{L}}{\partial \dot q} = m\dot q$$

and rewrite E as a function of (q, p) via

$$H(q,p) = \frac{|p|^2}{2} + V(q).$$

Then Newton's second law is equivalent to

$$\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q},$$

which is a first-order system on the phase space $R^3 \times \mathbb{R}^{3}$[3].

[3] Note that for time-dependent Lagrangians, we get the additional condition

$$\frac{\partial}{\partial t}H = -\frac{\partial}{\partial t}\mathcal{L}.$$

Definition 3.1. Given a function $H = H(q,p) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$, we associate to H the *Hamiltonian system*

$$\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q}. \tag{3.2}$$

The space $\mathbb{R}^d \times \mathbb{R}^d$ is called the *phase space.*

Observe that we can rewrite the Hamiltonian system (3.2) by introducing the (symplectic) matrix

$$\mathbb{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_{d\times d} \\ -\mathbf{1}_{d\times d} & \mathbf{0} \end{pmatrix},$$

so that (3.2) becomes

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \mathbb{J}\nabla H(q,p) =: X_H(q,p).$$

So, we can think of Hamiltonian system also to be the flow with respect to the vector field X_H on the space $\mathbb{R}^d \times \mathbb{R}^d$. We shall explore this symplectic structure more after we have discussed a few simple properties of Hamiltonian systems. Notice that we will restrict the discussion to Hamiltonian system on linear symplectic spaces (such as $\mathbb{R}^d \times \mathbb{R}^d$ with the symplectic matric $\mathbb{J}$, or as we will also shortly see, function spaces). In general, one can study Hamiltonian systems on manifolds that carry a simplectic structure, that is manifolds that look locally like a symplectic linear space.

We start by showing that H is a constant of motion.

Proposition 3.2. *Given a solution* (q, p) *to the Hamiltonian system* (3.2), *it holds*

$$\frac{d}{dt} H(q, p) = 0.$$

Proof. It holds

$$\begin{aligned} \frac{d}{dt} H(q(t), p(t)) &= \frac{\partial H}{\partial q}(q, p) \cdot \dot{q} + \frac{\partial H}{\partial p} \cdot \dot{p} \\ &= -\dot{p} \cdot \dot{q} + \dot{q} \cdot \dot{p} = 0. \end{aligned}$$

□

Constants of motion do play an important role in studying Hamiltonian systems. Consider the following example.

Example 3.3. Consider a one-dimensional pendulum of length ℓ. Then $q\colon [0, t_0] \to \mathbb{R}$ denotes the displacement of the angle from equilibrium position . Easy trigonometry tells us that the forces acting on the pendulum are given by

$$F(\theta) = -mg \sin(q).$$

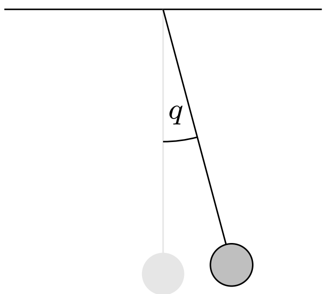

Fig. 3.1: One-dimensional pendulum.

Since the acceleration depends on the length, we have $a = \ell \ddot{q}$, so that we obtain the equation of motion from Newton's second law

$$m\ddot{q} = -\frac{mg}{\ell} \sin(q).$$

We easily verify that this is a Hamiltonian system given the Hamiltonian

$$H(q, p) = \frac{1}{2m}|p|^2 - m\frac{g}{\ell} \cos(q).$$

Proposition 3.2 tells us that solutions lie on level sets of the Hamiltonian. The level sets of a Hamiltonian are usually represented in a phase portrait.

We observe that for small energies, the trajectories in phase space are closed curves: we obtain periodic motion and the pendulum just swings back and forth. Reaching a certain critical energy, we see a change in the dynamics. The trajectory signifying this change is called *separatrix*. For higher energies, we see rotations in the phase portrait and the pendulum swings round and round.

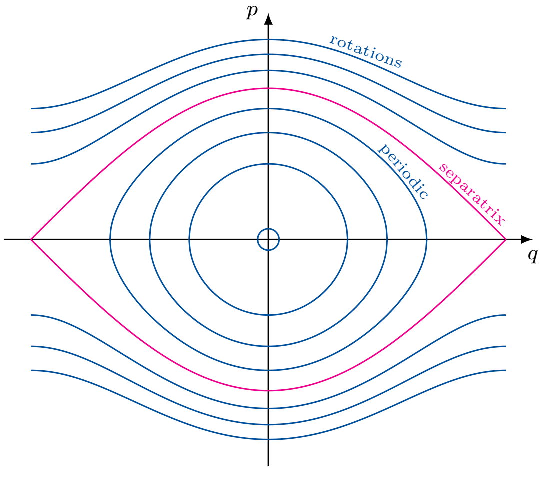

Fig. 3.2: Phase portrait of the Hamiltonian $H(q, p) = \frac{1}{2m}|p|^2 - \frac{mg}{\ell} \cos(q)$.

Observe that for two-dimensional phase spaces (that is a system with one-dimensional Lagrangian state space), this immediately implies that all Hamiltonian systems are integrable. The level sets of the Hamiltonian give us exactly the manifolds on which the motion lies. For higher-dimensional Hamiltonians we need more integrals of motion to obtain complete integrability, as we will discuss in Section 2.

Exercise. Draw the phase portrait for the simple harmonic oscillator discussed in Example 1.2.

We now turn to the symplectic structure of Hamiltonian systems. Recall that we defined the matrix

$$\mathbb{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_{d \times d} \\ -\mathbf{1}_{d \times d} & \mathbf{0} \end{pmatrix}.$$

This matrix gives rise to an anti-symmetric, bilinear form on $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$ that we will call

$$\omega(\xi_1, \xi_2) = \xi_1 \mathbb{J} \xi_2 = \xi_1^q \cdot \xi_2^p - \xi_1^p \cdot \xi_2^q, \quad \xi_i = (\xi_i^q, \xi_i^p),\ i = 1, 2.$$

Definition 3.4. A linear map $A\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is called *symplectic* if

$$A^T \mathbb{J} A = \mathbb{J}, \tag{3.3}$$

or, equivalently, if $\omega(A\xi_1, A\xi_2) = \omega(\xi_1, \xi_2)$ holds for every $\xi_1, \xi_2 \in \mathbb{R}^{2d}$.

Note that since $\det(\mathbb{J}) = 1$, symplectic maps preserve the area:

$$1 = \det(\mathbb{J}) = \det(A^T \mathbb{J} A) = \det(A)^2.$$

A general nonlinear map is called symplectic if it looks locally like a symplectic linear map.

Definition 3.5. Let $U \subset \mathbb{R}^{2d}$ be an open set and $g\colon U \to \mathbb{R}^{2d}$ be C^1. Then we call g *symplectic* if its Jacobian $Dg(q,p)$ is symplectic for every $(q,p) \in U$, i.e. if

$$Dg(q,p)^T \mathbb{J} Dg(q,p) = \mathbb{J}, \quad (q,p) \in U. \tag{3.4}$$

Liouville observed that will regions in the phase space change their shape under the flow of a Hamiltonian system, their volume is conserved. We give a proof using that flows of Hamiltonian systems are symplectic.

Definition 3.6. The *phase flow* of a Hamiltonian system H on $\mathbb{R}^d \times \mathbb{R}^d$ is defined by

$$\Phi_t\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d : (q_0, p_0) \mapsto (q(t), p(t)),$$

where (q,p) is the solution to the Hamiltonian system given the Hamiltonian H.

Exercise. Verify that $(\Phi_t)_{t\in\mathbb{R}}$ is a group.

Since we are talking about the flow, it is a good approach to figure out which vector fields generate Hamiltonian system. Note that given a Hamiltonian H, we associated the vector field

$$X_H = \mathbb{J} \nabla H.$$

Then the Hamiltonian system with Hamiltonian H is the same as the flow w.r.t. the vector field

$$\frac{d}{dt}(p,q) = X_H(p,q).$$

This explains the following definition.

Definition 3.7. A vector field $X\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is called *Hamiltonian* if there is a C^1-Hamiltonian map $H\colon \mathbb{R}^{2d} \to \mathbb{R}$ such that

$$X(p,q) = \mathbb{J}^{-1} \nabla H.$$

We can now give a characterisation of Hamiltonian vector fields.

Proposition 3.8. *Let $X : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be a vector field. Then X is a Hamiltonian vector field if and only if $DX(q, p)$ is ω-skew for all $(q, p) \in \mathbb{R}^{2d}$, that is*

$$\omega(DX(q,p)\xi_1, \xi_2) = -\omega(\xi_1, DX(q,p)\xi_2), \quad \textit{for all } \xi_1, \xi_2 \in \mathbb{R}^{2d}. \tag{3.5}$$

Proof. Assume X is a Hamiltonian vector field, that is $X = \mathbb{J}\nabla H$. By definition of ω, it must hold

$$\omega(X(q,p), \xi_2) = (\nabla H(q,p))^T \mathbb{J}^T \mathbb{J}\xi_2 = -dH(q,p)\xi_2$$

Differentiating this relation in direction ξ_1, one obtains

$$\omega(DX(q,p)\xi_1, \xi_2) = -D^2H(q,p)(\xi_1, \xi_2) = \omega(DX(q,p)\xi_2, \xi_1)$$

by symmetry of $D^2H(q,p)$. Hence, $DX(q,p)$ must be ω-skew.

Vice versa, assume that $DX(q,p)$ is ω-skew, then define

$$H(q,p) = \int_0^1 \omega(X(t(q,p)), (q,p))\,\mathrm{d}t + \text{const.}$$

Then

$$\begin{aligned} dH(q,p)\cdot\xi &= \int_0^1 \omega(DX(t(q,p))t\xi, (q,p)) + \omega(X(t(p,q)), \xi)\,\mathrm{d}t \\ &= \int_0^1 t\omega(DX(t(q,p))(q,p), \xi) + \omega(X(t(p,q)), \xi)\,\mathrm{d}t \\ &= \omega\left(\int_0^1 \frac{d}{dt}[tX(t(q,p))]\,\mathrm{d}t, \xi\right) \\ &= (X(q,p), \xi) \end{aligned}$$

for all ξ. Hence $\nabla H = \mathbb{J}^{-1}X$. □

Theorem 3.9. *Let H be a twice continuously differentiable Hamiltonian on $\mathbb{R}^{2d}$. Then, for each fixed t, Φ_t is a symplectic map.*

Proof. Φ_t is also the flow of the vector field X_H, that is

$$\frac{d}{dt}\Phi_t(q,p) = X_H(\Phi_t(q,p))$$

In particular, it holds

$$\frac{d}{dt}\left[D\Phi_t(q,p)\xi\right] = D\left[\frac{d}{dt}\Phi_t(q,p)\right]\xi = DX_H(\Phi_t(q,p))D\Phi_t(q,p)\xi.$$

Hence, we obtain

$$\begin{aligned} \frac{d}{dt}\omega(D\Phi_t(q,p)\xi_1, D\Phi_t(q,p)\xi_2) &= \omega(DX_H(q,p)[D\Phi_t(q,p)\xi_1], \xi_2) \\ &\quad + \omega(\xi_1, DX_H(q,p)[D\Phi_t(q,p)\xi_2]) \\ &= 0 \end{aligned}$$

since X_H is ω-skew by Proposition 3.8. But this implies that

$$\omega(D\Phi_t(q,p)\xi_1, D\Phi_t(q,p)\xi_2) = \omega(D\Phi_0(q,p)\xi_1, D\Phi_0(q,p)\xi_2) = \omega(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in \mathbb{R}^{2d}$ and $(q,p) \in \mathbb{R}^{2d}$. □

Corollary 3.10. *Let $U \subset \mathbb{R}^d \times \mathbb{R}^d$ be an open set in the phase space and H a twice continuously differentiable Hamiltonian. Then*

$$|\Phi_t(U)| = |U|.$$

Proof. This follows directly from the observation that since Φ_t is symplectic, it holds

$$|\det D\Phi_t(q,p)| = 1, \quad (q,p) \in \mathbb{R}^{2d}.$$

□

The converse to Theorem 3.9 is also true.

Theorem 3.11. *Let $X : U \to \mathbb{R}^{2d}$ be continuously differentiable vector field and consider the autonomous ordinary differential equation*

$$\dot{y} = X(y)$$

and its flow Φ_t. Then the system $\dot{y} = X(y)$ is locally Hamiltonian, that is for each point in space-time (t_0, q_0, p_0), there is a neighborhood $(t_0, q_0, p_0) \in V$ in space-time and a Hamiltonian H such that the flow agrees with the flow of the Hamiltonian system given H if and only if Φ_t is symplectic for all sufficiently small t.

Now that we have seen some first important properties of Hamiltonian systems, we turn to study examples and generalise the notions we have introduced. We start with a guiding example for the next part of the course, the n-body problem.

Example 3.12 (The n-body problem)**.** Consider n particles moving in an inertial system with positions $q_i : \mathbb{R} \to \mathbb{R}^3$, $i = 1, \dots, n$ and masses m_i, and assume that the only force acting on the motion is mutual gravitational attraction. Newton's law of gravity says that the magnitude of force on the i-th particle coming from particle j is proportional to the product of their masses and inversely proportional to the square of their distance.

That is, by Newton's second law we obtain the system of ODEs given by

$$m_i \ddot{q}_i = \sum_{j \neq i} \frac{g m_i m_j (q_i - q_j)}{|q_i - q_j|^3} = \frac{\partial U}{\partial q_i}, \tag{3.6}$$

where the potential U is given by

$$U(q_1, \dots, q_n) = \sum_{1 \leq i < j \leq n} \frac{g m_i m_j}{|q_i - q_j|}.$$

Then (3.6) is a Hamiltonian system on the phase space $R^{3n} \times \mathbb{R}^{3n}$ given the Hamiltonian

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} \frac{|p_i|^2}{2m_i} - U(q_1, \dots, q_n).$$

This problem consists of $6n$ equations and even for $n = 3$ solving these would require a large number of integrals. As we will discuss later, we expect resonating solutions for the three-body problem.

The situation for $n = 2$ is much better, though.

Example 3.13 (The 2-body problem)**.** Consider the following change of coordinates[4]

$$\bar{q} = \mu_1 q_1 + \mu_2 q_2, \quad p = p_1 + p_2, \quad u = q_2 - q_1, \quad v = -\nu_2 p_1 + \nu_1 p_2.$$

Here,

$$\nu_1 = \frac{m_1}{m_1 + m_2}, \quad \nu_2 = \frac{m_2}{m_1 + m_2}, \quad \nu = m_1 + m_2, \quad M = \frac{m_1 m_2}{m_1 + m_2}.$$

There is a natural interpretation for this change of coordinates: $\bar{q}$ is the center of mass, p is the total momentum, u is the relative position. In the new coordinates, the Hamiltonian takes the form

$$H(\bar{q}, u, p, v) = \frac{|p|^2}{2\nu} + \frac{|v|^2}{2M} - \frac{m_1 m_2}{|u|}.$$

We can now determine the new equations of motions and obtain

$$\begin{aligned} \dot{\bar{q}} &= \frac{\partial H}{\partial p} = \frac{p}{\nu}, & \dot{p} &= -\frac{\partial H}{\partial \bar{q}} = 0, \\ \dot{u} &= \frac{\partial H}{\partial v} = \frac{v}{M}, & \dot{v} &= -\frac{\partial H}{\partial u} = -\frac{m_1 m_2 u}{|u|^3} \end{aligned}$$

This proves that the combined linear momentum p is an integral[5] of the dynamical system and hence constant and the center of mass $\bar{q}$ moves on a straight line given the total momentum p. Hence, the system reduces to an equation for (u, v) of the form

$$\ddot{u} = \frac{g\nu u}{|u|^3}.$$

This is a central-force problem, also known as the Kepler problem.

Exercise. Solve the Kepler problem.

1.2 Poisson brackets

Consider a function $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined on the phase space. Then, if (q, p) is the flow of a Hamiltonian system with Hamiltonian H, it holds

$$\begin{aligned} \frac{d}{dt} f(q(t), p(t)) &= \frac{\partial f}{\partial q}(q(t), p(t)) \cdot \dot{q}(t) + \frac{\partial f}{\partial p}(q(t), p(t)) \cdot \dot{p}(t) \\ &= \frac{\partial f}{\partial q}(q(t), p(t)) \cdot \frac{\partial H}{\partial p}(q(t), p(t)) - \frac{\partial f}{\partial p}(q(t), p(t)) \cdot \frac{\partial H}{\partial q}(q(t), p(t)). \end{aligned}$$

This motivates the following definition.

Definition 3.14 (Poisson bracket)**.** Let $f, g\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be differentiable functions. Then we define the *Poisson bracket* via

$$\{f, g\} = \sum_{i=1}^{d} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) = \nabla f \cdot \mathbb{J} \nabla g.$$

The following properties follow immediately from the definition.

[4] We will discuss the general theory on changes of coordinates that keep the Hamiltonian structure of the system later.

[5] Such a coordinate is also called cyclic.

Lemma 3.15. *Let $f, g, h\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be differentiable functions*

1) ***Bilinearity:***

$$\{\alpha_1 f + \alpha_2 g, h\} = \alpha_1\{f, h\} + \alpha_2\{g, h\}$$

and likewise for the second component;

2) ***Anti-commutativity:***

$$\{f, g\} = -\{g, f\}.$$

In particular, $\{f, f\} = 0$;

3) ***Jacobi identity:***

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

4) ***Product identity:***

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Example 3.16. The canonical coordinates $q = (q_1, \ldots, q_d)$, $p = (p_1, \ldots, p_d)$ satisfy the following relations for their Poisson brackets[6]

$$\{q_i, q_j\} = 0,$$
$$\{p_i, p_j\} = 0,$$
$$\{q_i, p_j\} = \delta_{ij}.$$

[6] Notice the similarity to quantum mechanics for the commutation relations of the position and momentum operator. This is not a coincidence.

Example 3.17. We can rewrite the Hamiltonian equations as

$$\dot{q} = \{q, H\},$$
$$\dot{p} = \{p, H\}.$$

Corollary 3.18. *Let $f\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $(q, p) = (q(t), p(t))$ the flow of a Hamiltonian system with Hamiltonian H, then*

$$\frac{d}{dt} f(q, p) = \{f, H\}(q, p).$$

In particular, f is a constant of motion if and only if

$$\{f, H\} = 0.$$

We say that two functions f and g are in *involution* if

$$\{f, g\} = 0.$$

Lemma 3.19. *Let $f, g\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $(q(t), p(t))$ the flow of a Hamiltonian system with Hamiltonian H, then*

$$\frac{d}{dt}\{f(q,p), g(q,p)\} = \left\{\frac{d}{dt} f(q,p), g(q,p)\right\} + \left\{f(q,p), \frac{d}{dt} g(q,p)\right\}.$$

In particular, the Poisson bracket of two constants of motion is again a constant of motion.

Exercise. Let $d = 3$ and define $L = q \times p$ the usual cross product. Compute

$$\{L_i, L_j\}, \quad i, j = 1, 2, 3.$$

Conclude that if two components of the angular momentum are constants of motion, then so must be the third.

1.3 Canonical transformations and generating functions

Classically, physical systems are independent of the change of observer, that is we can change our coordinate system. In Lagrangian dynamic, where the fundamental object is the evolution of our particle q, we can find a change of variables $Q = F(q)$ by any diffeomorphism F and rewrite the Lagrangian.

In Hamiltonian dynamics, we are working on the phase space $\mathbb{R}^d \times \mathbb{R}^d$ and our corresponding ODE consists of $2d$-components. This gives us more flexibility in that we can find changes of variables that mix q and p. But not every possible change of coordinates will keep the Hamiltonian structure.

Define a general change of coordinates by

$$(Q, P)\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d : (q, p) \mapsto (Q(q, p), P(q, p))$$

and denote its Jacobian $D(P, Q) = \mathcal{J}$. Then, we have

$$\frac{d}{dt}Q_i(q,p) = \{Q_i, H\}(q,p) = \nabla Q_i \mathbb{J} \nabla_{(q,p)} H(q,p) = \nabla Q_i \cdot \mathbb{J}(D(Q,P))^T \nabla_{(Q,P)} H,$$
$$\frac{d}{dt}P_i(q,p) = \{P_i, H\}(q,p) = \nabla P_i \mathbb{J} \nabla_{(q,p)} H(q,p) = \nabla P_i \cdot \mathbb{J}(D(Q,P))^T \nabla_{(Q,P)} H.$$

In total, this demonstrates

$$\frac{d}{dt}(Q, P) = D(Q,P)\mathbb{J}(D(Q,P))^T \nabla_{(Q,P)} H$$

and the Hamiltonian structure is invariant if and only if the change of coordinates is symplectic

$$(D(Q,P))\mathbb{J}D(Q,P)^T = \mathbb{J}.$$

Definition 3.20. A change of coordinates $(q, p) \mapsto (Q, P)$ with Jacobian $\mathcal{J} = D(Q, P)$ is called a *canonical transformation* if

$$\mathcal{J}\mathbb{J}\mathcal{J}^T = \mathbb{J}.$$

Proposition 3.21. *The Poisson bracket is invariant under canonical transformations. Conversely, any transformation $(q, p) \mapsto (Q, P)$ which preserves the Poisson bracket so that*

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \quad \textit{and} \quad \{Q_i, P_j\} = \delta_{ij}$$

is canonical.

Proof. Let $f, g\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ two differentiable functions and denote by $\mathcal{J} = D(Q, P)$ the Jacobian of the canonical transformation. Denote by $y = (q, p)$ and $Y = (Q, P)$

$$\frac{\partial f}{\partial y_i} = \sum_{j=1}^{2d} \frac{\partial f}{\partial Y_j} \mathcal{J}_{ji}.$$

Hence,

$$\begin{aligned}\{f,g\} = \nabla_{(q,p)} f \cdot \mathbb{J} \nabla_{(q,p)} g &= \sum_{i,k=1}^{2d} \frac{\partial f}{\partial y_i} \mathbb{J}_{ik} \frac{\partial g}{\partial y_k} \\ &= \sum_{i,j,k,l=1}^{2d} \frac{\partial f}{\partial Y_j} \mathcal{J}_{ji} \mathbb{J}_{ik} \mathcal{J}_{lk} \frac{\partial f}{\partial Y_l} \\ &= \sum_{j,l=1}^{2d} \frac{\partial f}{\partial Y_j} \mathbb{J}_{jl} \frac{\partial f}{\partial Y_l}.\end{aligned}$$

For the converse, note that for a general transformation, we have

$$\mathcal{J} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}.$$

But then, we can compute

$$\mathcal{J} \mathbb{J} \mathcal{J}^T = \begin{pmatrix} \{Q_i, Q_j\} & \{Q_i, P_j\} \\ \{P_i, Q_j\} & \{P_i, P_j\} \end{pmatrix}.$$

So whenever the canonical Poisson bracket relations are satisfied, the transformation is canonical. □

In certain situations, there is a simple method of constructing a canonical transformation between the coordinates (q,p) and (Q,P). Assume we are given a function $F(q,P)$ depending on the old positions and new momenta[7] such that

$$\det \left(\frac{\partial^2 F}{\partial q_i \partial P_j} \right)_{ij} \neq 0.$$

[7] This might look very strange at first sight, but it will come in very useful in the next section.

Then we may construct a canonical transformation via

$$p_i = \frac{\partial F}{\partial q_i} \quad \text{and} \quad Q_i = \frac{\partial F}{\partial P_i}.$$

Definition 3.22. The function F is called the *generating function* of the canonical transformation.

Proposition 3.23. *The change of coordinates given by a generating function $F = F(q,P)$ with*

$$\det \left(\frac{\partial^2 F}{\partial q_i \partial P_j} \right)_{ij} \neq 0$$

defined by

$$p_i = \frac{\partial F}{\partial q_i} \quad \textit{and} \quad Q_i = \frac{\partial F}{\partial P_i}.$$

is a canonical transformation.

Proof. By Proposition 3.21 it is enough to study the Poisson brackets. Note that

$$\frac{\partial p}{\partial P} = \frac{\partial^2 F}{\partial q \partial P} = \frac{\partial Q}{\partial q}.$$

Inserting this into the Poisson brackets gives us the canonical relations: it is

$$\begin{aligned}\{Q_i, P_j\}_{i,j} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} \\ &= \frac{\partial^2 F}{\partial q \partial p}\left(\frac{\partial^2 F}{\partial q \partial P}\right)^{-1} \\ &= \mathrm{Id}\,.\end{aligned}$$

This proves the claim. □

2 *Integrability and the Arnold–Liouville theorem*

In this next section, we will define the notion of complete integrability for Hamiltonian. For this special class of Hamiltonian systems, we will construct a canonical transformation which makes the dynamics very easy to understand. This is known as the Arnold-Liouville theorem.

2.1 Complete integrability

Recall that we have seen that we can always find at least one integral of motion, that is the Hamiltonian H itself.

Definition 3.24 (Completely integrable Hamiltonian system)**.** A Hamiltonian system with Hamiltonian $H\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is *(completely) integrable* if it possesses d integrals of motion $f_1 = H, f_2, \dots, f_d$ so that

(a) these integrals of motion are functionally independent, that is $\nabla f_1, \dots, \nabla f_d$ are linearly indendent on a dense, open subset of $\mathbb{R}^d \times \mathbb{R}^d$,

(b) they are pairwise in involution with respect to the Poisson bracket, i.e.

$$\{f_i, f_j\} = 0, \quad \text{for all } i, j \in \{1, \dots, d\},$$

(c) the vector fields $X_{f_i} = \mathbb{J}\nabla f_i$ are complete[8].

[8] A vector field is called *complete* if it generates a global flow.

Example 3.25. Let $d = 1$. Hamiltonian systems with Hamiltonian $H\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ so that $\nabla H(q,p) \neq 0$ on an open, dense subset are completely integrable with the choice $I_1 = H$.

2.2 Interlude: Lie group actions, and Noether's theorem

Before studying integrable Hamiltonian systems in more detail, we go back to the two-body problem and show that it is integrable.

Recall the following definition.

Definition 3.26 (Lie group and associated Lie algebra)**.**

1) A *Lie group* is a finite-dimensional smooth manifold G that carries a group structure for which the product map $G \times G \to G : (g,h) \mapsto g \cdot h$ and the inverse map $G \to G : g \mapsto g^{-1}$ are smooth.

2) Given a Lie group G, we associate the Lie algebra[9]

$$\mathfrak{g} = T_e G,$$

where e denotes the neutral element of the Lie group G.

[9] Recall that an algebra is called a Lie algebra if it carries a bilinear, anti-commutative form that satisfies the Jacobi identity, cf. Lemma 3.15.

Example 3.27.

(1) The group of symplectic matrices $Sp(d)$ is a Lie group with Lie algebra $\mathfrak{sp}(d)$.

(2) The group of orientation-preserving linear isometries $SO(d)$ is a Lie group with Lie algebra $\mathfrak{so}(d)$.

Proposition 3.28. *For each element $\xi \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of a Lie group G, there exists a unique function $f_\xi \colon \mathbb{R} \to G$ such that*

$$f_\xi(s+t) = f_\xi(s) f_\xi(t), \quad \textit{for all } s,t \in \mathbb{R}$$

and such that

$$f'_\xi(0) = \xi.$$

f_ξ is called exponential and we denote $f_\xi(t) = \exp(t\xi)$.[10]

[10] For a general Lie group G, we can define the map f_ξ as the flow of the vector field

$$X_\xi(g) = dL_g(e)(\xi) \in T_g G,$$

where $L_g(h) = g \cdot h$ is the left multiplication.

Example 3.29. In the case $G = Sp(d)$ or $G = SO(d)$, it holds $\exp(t\xi)$ is the usual matrix exponential.

Definition 3.30 (Actions of Lie groups)**.**

1) An action of a Lie group on a manifold M is a group homomorphism $\psi \colon G \to \mathrm{Diff}(M)$. We usually denote $g.m = \psi(g)(m)$.

2) If $M = \mathbb{R}^d \times \mathbb{R}^d$, we say that the action is symplectic if $\psi(g)$ is symplectic for every $g \in G$.

As we can see, the Lie groups[11] from the example act on the phase space. When the Lie group acts by symplectomorphisms, they keep the Hamiltonian structure of the flow.

[11] In our case of Hamiltonian systems on linear spaces, strictly speaking we don't need to use the terminology of Lie groups. We introduce the technology here anyway to help the reader generalise the notions to the case of symplectic manifolds. This will also be useful later.

Definition 3.31. Let ψ be a Lie group action on $\mathbb{R}^{2d}$. We define the map $\Psi \colon \mathfrak{g} \to C^\infty(\mathbb{R}^{2d}; \mathbb{R}^{2d})$ by

$$\Psi(\xi)(x) = \frac{d}{dt}\psi(\exp(t\xi), x)$$

for every $f \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

Definition 3.32 (Hamiltonian action)**.** We call a symplectic action *Hamiltonian* if $\Psi(\xi)$ is a Hamiltonian vector field for each $\xi \in \mathfrak{g}$. [12] We define the *co-moment map* $\Phi \colon \mathfrak{g} \to C^\infty(M)$ so that $\Psi(\xi) = \mathbb{J} X_{\Phi(\xi)}$.

[12] This definition is not without subleties. For once, we could have also defined a Hamiltonian action to be a group action of a Lie group G by Hamiltonian symplectomorphisms, that is by symplectomorphisms given by the flow w.r.t. a family of smooth Hamiltonians. In fact, if the first deRham-cohomology vanishes, and this is the case for our linear space, then there is no difference between an action of a Lie group by symplectomorphisms and a Hamiltonian action. This general fact from symplectic geometry is beyond the scope of this lecture.

Definition 3.33. A Lie group G is a symmetry group of a Hamiltonian system if there exists a Hamiltonian action ψ such that $H \circ \psi(g) = H$ for every $g \in G$.

Example 3.34. For $V\colon [0,\infty) \to \mathbb{R}$ consider the Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 - V(|q|).$$

Then $SO(3)$ is a symmetry group for the Hamiltonian system via the induced Hamiltonian action on $\mathbb{R}^d \times \mathbb{R}^d$ by acting on the first coordinate.

Theorem 3.35 (Noether's theorem)**.** *If G is a symmetry group of a Hamiltonian system and $\Phi\colon \mathfrak{g} \to C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is its co-moment map, then for every $\xi \in \mathfrak{g}$ the function $\Phi(\xi)$ is an integral of motion.*

Corollary 3.36. *Any Hamiltonian system of the form $H(q,p) = \frac{p^2}{2} + V(|q|)$ with a central force field is completely integrable. In particular, the Kepler problem, cf. Example 3.13, is completely integrable.*

Proof. Note that $\dim \mathfrak{so}(3) = 3$. Consider a basis ξ_1, ξ_2, ξ_3 of $\mathfrak{so}(3)$. Then $\Phi(\xi_1)$, $\Phi(\xi_2)$ and $\Phi(\xi_3)$ are integrals of motion. Note that since $(\mathbb{R}^3, \times)$ and $\mathfrak{so}(3)$ are isomorphic, we can identify $\Phi(\xi_i)$ with the angular momentum L_i. □

Theorem 3.37. *If $d = 3$, then any Hamiltonian system of the form $H(q,p) = \frac{|p|^2}{2} + V(|q|)$ with a central force field is completely integrable. In particular, the Kepler problem, cf. Example 3.13, is completely integrable.*

Proof. Consider $L = q \times p$. We have already seen that the canonical Poisson relations are given by $\{L_i, L_j\} = \varepsilon_{ijk} L_k$. Note that from Corollary 3.36 we conclude that $\{L_i, H\} = 0$. Furthermore, observe that

$$\{L_1^2 + L_2^2 + L_3^2, H\} = 0.$$

Now, we can see that the three constants of motion H, L_1 and $L_1^2 + L_2^2 + L_3^2$ are functionally independent to conlude the proof. □

There are a few more known examples of completely integrable systems.

Example 3.38.

(1) The harmonic oscillator

$$H(q,p) = \frac{i=1}{d}\frac{p_i^2}{2} + \frac{\omega_i q_i^2}{2}$$

is completely integrable.

(2) The planar movement of a body under gravitational attraction of two fixed central bodies

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2) - \frac{m_1}{r_1} - \frac{m_2}{r_2},$$

where r_1 and r_2 are the distance of the moving body from the two central bodies

$$r_1 = \sqrt{(q_1 + c)^2 + q_2^2}, \quad r_2 = \sqrt{(q_1 - c)^2 + q_2^2} \tag{3.7}$$

(3) Certain spinning tops, such as the Euler, Lagrange and Kovalevskaya spinning top are integrable, see [Aud99].

2.3 The Arnold-Liouville theorem

Definition 3.39 (Lagrangian submanifold). A submanifold M of $\mathbb{R}^d \times \mathbb{R}^d$ with the symplectic form ω is called *Lagrangian* of $\omega|_M \equiv 0$ and M is maximal with this property.

Remark 3.40. Note that Lagrangian submanifolds have dimension d.

Theorem 3.41 (Arnold–Liouville theorem). *Consider a completely integrable Hamiltonian system $f = (f_1, \ldots, f_d)$ with Hamiltonian $H = f_1$. For $c \in \mathbb{R}^d$ define the level surface*[13]

$$M_c = \{(q,p) \in \mathbb{R}^d \times \mathbb{R}^d : f(q,p) = c\}.$$

Assume that c is a regular value for f. Then

1) *If M_c is compact and connected, then it is a Lagrangian submanifold which is diffeomorphic to the torus $\mathbb{T}^d = S^1 \times \ldots \times S^1$.*

2) *There are coordinates $\theta = (\theta_1, \ldots, \theta_d)$ on this torus such that the dynamics of the Hamiltonian system are given by*

$$\dot{\theta}_i = \omega_i(I_1, \ldots, I_d), \quad i = 1, \ldots, d.$$

3) *In a neighborhood of this torus in $\mathbb{R}^d \times \mathbb{R}^d$ one can introduce* action-angle coordinates *(θ, I) such that the angles $(\theta_1, \ldots, \theta_d)$ form a coordinate system for M_c and the actions $(I_1, \ldots, I_d)$ are first integrals.*

4) *The change of coordinates $(q,p) \mapsto (\theta, I)$ is canonical and in the new coordinates, the dynamic is given by*

$$\dot{\theta}_i = \omega_i(I_1, \ldots, I_d), \quad i = 1, \ldots, d,$$
$$\dot{I}_i = 0, \quad i = 1, \ldots, d.$$

The proof is lengthy and not without technicalities. We split the proof into two parts.

[13] Note that we know that solutions of the Hamiltonian system stay on M_c.

2.4 The geometry of M_c and angle variables

For the first part of the proof, we need some algebra.

Lemma 3.42. *Let $\{0\} \neq \Gamma \subset \mathbb{R}^d$ be a discrete additive subgroup. Then there exists $1 \leq m \leq d$ and m linear independent vectors $v_1, \ldots, v_m \in \mathbb{R}^d$ such that*

$$\Gamma = \bigoplus_{k=1}^{m} \mathbb{Z} v_k.$$

The following result is a generalisation of the orbit-stabiliser theorem for actions of groups on a set to the situation of the action of a Lie group on a manifold, see e.g. [BD95, Prop. 4.6].

Proposition 3.43 (Orbit-stabiliser theorem for Lie groups). *Let M be a manifold equipped with a smooth and transitive action of a Lie group G. Denote for $m \in M$ the stabiliser*

$$\mathrm{Stab}(m) = \{g \in G : g.m = m\}.$$

Then the spaces M and $G/\mathrm{Stab}(m)$ are diffeomorphic.

And we also need some differential geometry. By the inverse function theorem and from the condition that the functions $f_1, \dots, f_d$ are functionally independent, we may already conclude that for any regular value $c \in \mathbb{R}^d$ for f, the space M_c is a manifold. Since $X_{f_1}, \dots, X_{f_d}$ forms a basis of the tangent space, this manifold is d-dimensional.

Lemma 3.44. *Define $X_i := X_{f_i} = \mathbb{J}\nabla f$. Then the X_i are vector fields on M such that $[X_i, Y_j] = 0$ for all $i, j = 1, \dots, d$.*

Theorem 3.45. *Let M be a manifold and $X, Y \colon M \to TM$ vector fields with flows Φ_t^X and Φ_t^Y. Then $[X, Y] = 0$ if and only if their flows commute, that is*

$$\Phi_t^X \circ \Phi_s^Y = \Phi_s^Y \circ \Phi_t^X \quad \textit{for all } t, s.$$

We split the proof of Theorem 3.41 into two parts. First, we show that, for any regular value c of f, M_c is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{d-k}$.

Proposition 3.46. *Let M be an d-dimensional equipped with d linearly independent, complete vector fields $X_1, \dots, X_d$ so that*

$$[X_i, X_j] := X_i \circ X_j - X_j \circ X_i = 0.$$

Then there exists $k \in \{0, \dots, d\}$ and a diffeomorphism

$$M \cong \mathbb{T}^k \times \mathbb{R}^{d-k}. \tag{3.8}$$

Proof. Denote by Φ_t^k the flow of the vector field X_k. Since $[X_i, X_j] = 0$, Theorem 3.45 implies that the flows commute. This allows us to define a transitive action of $\mathbb{R}^d$ on M in the following way:

$$\psi \colon \mathbb{R}^d \times M \to M : ((t_1, \dots, t_d), m) \mapsto \Phi_{t_1}^1 \circ \dots \circ \Phi_{t_d}^d(m).$$

Now, fix $m_0 \in M$ and define

$$\Gamma = \{t \in \mathbb{R}^d : \psi(t, m_0) = m_0\} = \mathrm{Stab}(m_0).$$

Observe that since the vector fields are complete and span the tangent space at every point, this gives a transitive action. Then by Proposition 3.43, we already know that $M \cong \mathbb{R}^d/\Gamma$. It remains to prove that Γ is discrete and does not depend on the choice of m_0.

Let $m_1 \in M$ be another point. Since the action is transitive, there is $r \in \mathbb{R}^d$ such that $\psi(r, m_0) = m_1$. Let $t \in \Gamma$, then $\psi(t, m_1) = \psi(t, \psi(r, m_0)) = \psi(r + t, m_0) = \psi(r, \psi(t, m_0)) = m_1$.

To show that the subgroup is discrete, notice that the map $\psi_{m_0} = \psi(\cdot, m_0) \colon \mathbb{R}^d \to M$ is, by construction, a local chart of M at m_0, that is there are neighborhoods $0 \in V \subset \mathbb{R}^d$ and $m_0 \in U \subset M$ so that $\psi_{m_0} \colon V \to U$ is a diffeomorphism. But since $\psi_{m_0}(0) = m_0$, this means that $t = 0$ is the only element in $\Gamma \cap V$. Arguing in the same way for any other $t \in \Gamma$ shows that Γ is discrete.

Now using Lemma 3.42 we may conclude that $\Gamma \cong \mathbb{Z}^k$ for some $0 \leq k \leq d$. After lifting this isomorphism to an isormophism of $\mathbb{R}^d$, we may conclude that $\mathbb{R}^d/\Gamma \cong \mathbb{R}^d/\mathbb{Z}^k \cong \mathbb{T}^k \times \mathbb{R}^{n-k}$, hence concluding the proof. $\square$

We have actually already constructed the angle variables. Therefore, observe that we have constructed a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^d & \xrightarrow{A\colon (\phi,y)\mapsto(t_1,\dots,t_d)} & \mathbb{R}^d \\ {\scriptstyle p\colon (\phi,y)\mapsto(\phi \bmod \mathbb{Z},y)}\big\downarrow & & \big\downarrow{\scriptstyle (t_1,\dots,t_d)\mapsto\psi(t,m_0)} \\ \mathbb{T}^k \times \mathbb{R}^{d-k} & \xrightarrow{\tilde{A}} & M_c \end{array}$$

We can use this commutative diagram to see that we have constructed the angle coordinates.

Corollary 3.47. *If M_c is compact and connected, then is is diffeomorphic to the torus $\mathbb{T}^d$. Furthermore, on the torus there are coordinates $\theta_1, \dots, \theta_d$ such that the phase flow of the Hamiltonian $H = f_1$ is a periodic motion in these coordinates, that is*

$$\dot{\theta} = \omega(f).$$

Proof. By Proposition 3.46, we have already seen that $M_c \cong \mathbb{T}^k \times \mathbb{R}^{d-k}$. But if M_c is compact, then $k = d$ and $M_c \cong \mathbb{T}^d$. Now consider the flow of the Hamiltonian Φ_t^1 and let $x \in M_c$. Since $\psi(\cdot, m_0)$ is surjective, there is $r \in \mathbb{R}^d$ such that $\psi(r, m_0) = x$. Hence,

$$\begin{aligned} \Phi_t^1(x) &= \psi(te_1, x) = \psi(te_1 + r, m_0) = \psi \circ A(tA^{-1}e_1 + A^{-1}r, m_0) \\ &= \tilde{A} \circ p(tA^{-1}e_1 + A^{-1}r). \end{aligned}$$

Hence, under the flow of A the coordinates $\theta = (\theta_1, \dots, \theta_d)$ satisfy $\theta(t) = \omega t + \theta(0)$ with $\omega = A^{-1}e_1$ and $\theta(0) = A^{-1}r$. This concludes the proof. ☐

2.5 The action variables

In the previous subsection, we have seen that $M_c \cong \mathbb{T}^d$ and the dynamics of the Hamiltonian system on the torus is conditionally periodic. In order to obtain a global understanding of the integrable Hamiltonian system, we want to understand the dynamics on all tori simultaneously. Therefore, note that for each value c, at least as long as M_c remains compact, we obtain that M_c looks like a torus. Stated differently, the phase space is foliated by tori.

Indeed, a neighborhood of M_c in phase space[14] topologically looks like a product of $\mathbb{T}^d$ and an open ball in $\mathbb{R}^d$. Considering the vector $f = (f_1, \dots, f_d)$ of our integrals, the dynamics in this neighborhood are very simple

[14] Note that the set of regular points is open.

$$\frac{d}{dt} f = 0, \quad \frac{d}{dt}\theta = \omega(f).$$

Unfortunately, the change of coordinates $(q, p) \mapsto (\theta, f)$ is not canonical. In order to remedy this problem, we need to redefine the integrals. We start by discussing this in the case of $d = 1$ and $f_1 = H$.

Example 3.48 (Action coordinates in $d = 1$)**.** Consider the Hamiltonian given by

$$H(q, p) = \frac{p^2}{2} + V(q).$$

Recall that since H is a constant of motion and $\nabla H \neq 0$ if $(p, q) \neq 0$, these Hamiltonian systems are always integrable, so the manifold M_c is given by the level sets of the Hamiltonian. These level sets look like tori in phase space and are parametrised by θ. In order to find the correct choice for I, we will look for a canonical transformation $(q, p) \mapsto (\theta, I)$, where in a neighborhood of M_c, we have

$$I = I(c), \quad \int_{M_c} \mathrm{d}\theta = 2\pi.$$

In order to construct the canonical transformation, we look for a generating functoin $F(q, I)$. Then, we need

$$p = \frac{\partial F}{\partial q}, \quad \theta = \frac{\partial F}{\partial I}, \quad H\left(q, \frac{\partial F}{\partial q}\right) = \tilde{H}(I).$$

Locally around M_c, we may write p as a function of q and c

$$p = p(q, c) = \sqrt{2(c - V(q))}.$$

Then the first condition for our generating function motivates

$$F(q, I) = \frac{1}{2\pi} \int_{[x_0, x]} p(q, c)\, \mathrm{d}q$$

where, for fixed $x_0 \in M_c = M_{h(I)}$ $[x_0, x]$ is a curve connecting x_0 and x on M_c and

$$I = I(c) = \frac{1}{2\pi} \int_{\gamma_c} p(q, c)\, \mathrm{d}q,$$

where γ_c is one cycle in $\mathbb{T} \times \{c\}$. Since we know that $p = \dot{q}$, we also find if the orbit has period ω that

$$\begin{aligned}
\frac{2\pi}{\omega} &= \int_{\gamma_c} \mathrm{d}t \\
&= \int_{\gamma_c} \frac{1}{p(q, c)}\, \mathrm{d}q \\
&= \int_{\gamma_c} \frac{d}{dc} \sqrt{2(c - V(q))}\, \mathrm{d}q \\
&= \frac{d}{dc} \int_{\gamma_c} \sqrt{2(c - V(q))}\, \mathrm{d}q.
\end{aligned}$$

This last step needs explanation since γ_c also depends on c. But if we change c to $c + \Delta c$, then we get a term of lower order since both the region of integration changes by order Δc and the integrand changes. In total, we find

$$\frac{2\pi}{\omega} = 2\pi \frac{dI}{dc}.$$

Next, note that we may then invert the relationship between I and c and write $c = c(I)$. This gives us the definition

$$F(q, I) = \int_{[q_0, q]} p(q, c(I))\, \mathrm{d}q,$$

where $[q_0, q]$ is a segment on $M_c = M_{c(I)}$. Note that we get

$$\frac{\partial F}{\partial q} = p$$

for free. To determine $\frac{\partial F}{\partial I}$, observe that after one full cycle, we land in $\gamma_c + [q_0, q]$ and find

$$F(q, I) = \int_{\gamma_c + [q_0, q]} p(q, c) \, \mathrm{d}q = 2\pi I + F(q, I),$$

so the function F is multi-valued which corresponds to θ being an angle. But now, this gives

$$\frac{\partial F}{\partial I} = \theta.$$

Finally, since the definition of I only depends on c, this also proves that $\dot{I} = 0$. This concludes the construction.

Exercise. Find S and I for the harmonic oscillator.

Now, we may generalise this construction to higher-dimensional integrable Hamiltonian systems in the following way. Assume that we have

$$\det\left(\frac{\partial f}{\partial p_j}\right) \neq 0. \tag{3.9}$$

Since $M_c \cong \mathbb{T}^d$, let $\gamma_1, \ldots, \gamma_d$ denote the cycles, i.e. in the coordinates on $\mathbb{T}^d$, we have e.g. $\gamma_j = \{0\}^{j-1} \times S^1 \times \{0\}^{d-j}$.

Definition 3.49. Consider an integrable Hamiltonian system on $\mathbb{R}^d$ with corresponding integrals $f_1, \ldots, f_d$. For $j = 1, \ldots, d$, the numbers

$$I_j = I_j(f_1, \ldots, f_d) = \frac{1}{2\pi} \int_{\gamma_j} p(q, c) \, \mathrm{d}q$$

are called *action variables*.

We need to show that the definitions are independent of the exact parametrisation of the torus.

Lemma 3.50. *Let $\tilde{\gamma}_j$ denote any other cycle in direction θ_j of the torus and denote*

$$\tilde{I}_j = \frac{1}{2\pi} \int_{\tilde{\gamma}_j} p(q, c) \cdot \, \mathrm{d}q.$$

Then $I_j = \tilde{I}_j$.

Proof. Observe that since on M_c it holds $f_i(q, p(q, c)) = c_i$, we may differentiate by q_k and obtain

$$\frac{\partial f_i}{\partial q_k} + \sum_{l=1}^{d} \frac{\partial f_i}{\partial p_l} \frac{\partial p_l}{\partial q_k} = 0.$$

Using that f_i and f_j are in involution, we obtain

$$\begin{aligned}
0 &= \{f_i, f_j\} \\
&= \sum_{k=1}^{d} \left(\frac{\partial f_i}{\partial q_k} \frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k} \frac{\partial f_j}{\partial q_k} \right) \\
&= \sum_{k,l=1}^{d} \left[\left(-\frac{\partial f_i}{\partial p_l} \frac{\partial p_l}{\partial q_k} \right) \frac{\partial f_j}{\partial p_k} - \frac{\partial f_i}{\partial p_k} \left(-\frac{\partial f_j}{\partial p_l} \frac{\partial p_l}{\partial q_k} \right) \right] \\
&= \sum_{k,l=1}^{d} \left[\left(-\frac{\partial f_i}{\partial p_k} \frac{\partial p_k}{\partial q_l} \right) \frac{\partial f_j}{\partial p_l} - \frac{\partial f_i}{\partial p_k} \left(-\frac{\partial f_j}{\partial p_l} \frac{\partial p_l}{\partial q_k} \right) \right] \\
&= \sum_{j,l=1}^{d} \frac{\partial f_i}{\partial p_k} \left(\frac{\partial p_l}{\partial q_k} - \frac{\partial p_k}{\partial q_l} \right) \frac{\partial f_j}{\partial p_l}.
\end{aligned}$$

Using (3.9) again, this implies that

$$\frac{\partial p_l}{\partial q_k} - \frac{\partial p_k}{\partial q_l} = 0.$$

First assume now that w.l.o.g.[15] $j = 1$ and $\gamma_j = S^1 \times \{0\}^{d-1}$ and $\tilde{\gamma} = S^1 \times \{\alpha\} \times \{0\}^{d-2}$. Now define A the area enclosed by the curves γ_1 and $\tilde{\gamma}_1$. Then by Green's theorem

[15] For the general case, just iterate this argument.

$$\begin{aligned}
I_1 - \tilde{I}_1 &= \int_{\gamma_1} p(q, c) \cdot \mathrm{d}q - \int_{\tilde{\gamma}_1} p(q, c) \cdot \mathrm{d}q \\
&= \int_{\partial A} p_1 \, \mathrm{d}q_1 + p_2 \, \mathrm{d}q_2 \\
&= \int_{A} \frac{\partial p_1}{\partial q_2} - \frac{\partial p_2}{\partial q_1} \, \mathrm{d}q_1 \, \mathrm{d}q_2 \\
&= 0.
\end{aligned}$$

This proves the claim. □

Now we are in the position to complete the complete proof of the Arnold-Liouville theorem.

Proof of Theorem 3.41. **Step 1.** We have seen in Corollary 3.47 that if M_c is compact, then it is diffeomorphic to the torus $\mathbb{T}^d$. Since M_c is d-dimensional and in order to prove that M_c is a Lagrangian submanifold, it suffices to prove that ω vanishes on $T_m M_c$ for every $m \in M_c$. But observe that the tangent space is spanned by the vector fields $X_1, \ldots, X_d$ and they satisfy

$$\omega(X_i(m), X_j(m)) = X_{f_i} \mathbb{J} X_{f_j} = 0.$$

Step 2. We have constructed the angular variables in Corollary 3.47.
Step 3. Since the action variables are only dependent on c and the dynamics takes place on the manifold M_c, we have $\dot{I} = 0$.
Step 4. We again need to check that the generating function

$$F(q, I) = \int_{[q_0, q]} p(q, c) \cdot \mathrm{d}q$$

satisfies

$$\frac{\partial F}{\partial q_i} = p_i, \quad \frac{\partial F}{\partial I_i} = \theta_i \quad \text{and} H\left(q, \frac{\partial F}{\partial q_i}\right) = \tilde{H}(I).$$

The first claim follows immediately from the definition and the third claim from Step 3. The second claim follows as in Example 3.48. □

Remark 3.51. In general, the manifold M_c ceases to exist for critical values of f. These critical values correspond to separatrices which separate phase space into different parts in which the dynamics might look slightly different. Compare this to the one-dimensional pendulum, where the separatrix divides the phase space into a part of oscillations and rotations, cf. Example 3.3.

The Arnold-Liouville theorem provides us with a complete understanding of the dynamics of integrable systems. From the initial condition, we determine $I = I_0$ and a corresponding torus on which the dynamics is conditionally periodic.

Definition 3.52. Let $\mathbb{T}^d$ be the d-dimensional torus and $\theta = (\theta_1, \dots, \theta_d) \mod 2\pi$ angular coordinates and $\omega = (\omega_1, \dots, \omega_d) \in [0, 2\pi)^d$. By a *conditionally periodic* motion we mean solutions to the differential equations

$$\dot{\theta} = \omega.$$

$\omega_1, \dots, \omega_d$ are called the *frequencies* of the conditionally periodic motioin. The frequencies are called *independent* if they are linearly independent over $\mathbb{Q}$.

Solutions to conditionally periodic motions are lines

$$\theta(t) = \theta_0 + \omega t$$

and these trajectories are called a *winding of the torus*.

Theorem 3.53. *If the frequencies are independent, then every trajectory is dense on the torus* $\mathbb{T}^d$.

Proof. The proof relies on ergodicity of the dynamical system. If we can prove that for any continuous function f on $\mathbb{T}^d$ and any initial condition θ_0 it holds

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\theta_0 + \omega t)\,\mathrm{d}t = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta)\,\mathrm{d}\theta, \tag{3.10}$$

then the theorem follows. Indeed, assume that there is a trajectory which is not dense. Then there is an open set U so that $\theta_0 + \omega t \notin U$ for every $t \geq 0$. But then, define a function f with spatial average equal to one and $f = 0$ outside of U to obtain a contradiction.

To prove (3.10), we approximate continuous functions by trigonometric polynomials

$$P(\theta) = \sum_{k\in\mathbb{Z}^d} a_k e^{ik\cdot\theta},$$

where only finitely many $a_k \neq 0$. Then, for $k \in \mathbb{Z}^d \setminus \{0\}$ and $f(\varphi) = e^{ik\cdot\varphi}$, we may compute

$$\frac{1}{T} \int_0^T f(\theta_0 + \omega t)\,\mathrm{d}t = \frac{1}{T} \int_0^T e^{ik\cdot\theta_0 + k\cdot\omega t}\,\mathrm{d}t = \frac{e^{ik\cdot\theta_0}}{ik\cdot\omega} \frac{e^{ik\cdot\omega T} - 1}{T} \longrightarrow 0$$

as $t \to \infty$ and

$$\int_{\mathbb{T}^d} e^{ik\cdot\theta}\,\mathrm{d}\theta = 0.$$

The case $k = 0$ follows immediately and the case of trigonometric polynomials follows from linearity of both sides. Approximating a continuous function by trigonometric polynomials gives us (3.10). □

3 KAM theory

4 The n-body problem and the solar system

5 Additional remarks and literature

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