1 Part One

1.1 Derivation of general thin-film equations

In this section we derive equation (0.2) from the two-dimensional, non-Newtonian Navier-Stokes system. We briefly run through the modelling of fluid dynamics by means of continuum mechanics to introduce the relevant objects and the fundamental relations between them. Afterwards we formulate *constitutive laws*, which encompass the structural properties of the liquids. More details on the derivation in the Newtonian case can be found in [Bat00].

1.1.1 The very basic situation

Picture the following situation: A body of fluid sits on top of a solid horizontal plate and evolves in time according to physical principles. We consider the situation in Euclidean space-time with two spatial components and one independent time component, so that we can talk about spatial and timely distances. Real world situations are described by our model if they are translation invariant in one direction. We introduce a coordinate system to the setup by fixing an arbitrary point on the surface to be the origin and then, for a given point P, we denote the distance to the plate as z and the distance between the origin and the orthogonal projection of P onto the plate as x. Similarly, we fix an arbitrary moment in time to be our reference and write t for the time passed since then.

Our primary assumption is that we can describe the domain where the liquid is situated at time t as the area enclosed by the graph of a function $h = h(t,x)$ and the solid plate. In other words, the liquid fills the open set $\Omega_t = \{(x,z) : 0 < z < h(t,x)\}$. As the fluid evolves with time, the free boundary $\{(x,z) : z = h(t,x)\}$ changes too and thus we have to consider h as a new dependent variable in our modelling. The precise behaviour of the liquid is described by continuum mechanics, based on a couple of basic conservation laws.

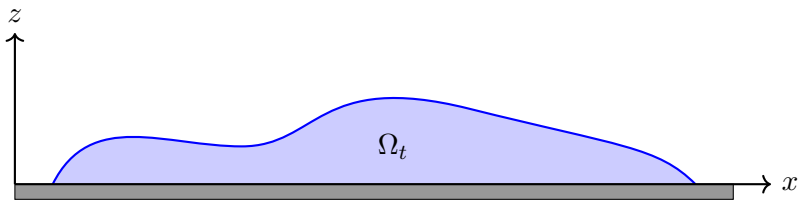

Figure 1.1: The set $\Omega_t = \{(x,z) : z = h(t,x)\}$.

1.1.2 From continuum mechanics to Navier-Stokes

In this framework, we imagine that our fluid extends into a set Ω_0 with varying density according to a mass distribution $\rho = \rho(t,x,z) \geq 0$. Necessarily, we have $\operatorname{supp}\rho(t,\cdot) = \Omega_t$. If we follow the path a particle takes as the fluid rearranges itself, we obtain a velocity field $U = U(t,x,z) = (u,v)$, which describes the momentary change of position in the horizontal direction with $u = u(t,x,z)$ and in the vertical direction with $v = v(t,x,z)$. Conversely, if such a velocity field is given, we can determine the position of a particle at time t, by integrating along the velocity field with respect to a given starting point (x_0,z_0). For future reference, we define $\Phi_t^U(x_0,z_0)$ at time t by solving

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^U(x_0,z_0) = U(t,\Phi_t^U(x_0,z_0)) \\ \Phi_0^U(x_0,z_0) = (x_0,z_0). \end{cases} \tag{1.1}$$

The variables U and ρ become interrelated by the basic assumptions of *conservation of mass* and *conservation of (angular) momentum*. In our modelling, this is enforced by the condition that the quantities

$$\begin{aligned} M(\omega_t) &:= \int_{\omega_t} \rho \,\mathrm{d}(x,z) \in \mathbb{R}, && \text{(Mass)} \\ P(\omega_t) &:= \int_{\omega_t} \rho\, U \,\mathrm{d}(x,z) \in \mathbb{R}^2, && \text{(Momentum)} \\ L(\omega_t) &:= \int_{\omega_t} \rho\, (-z,x)\cdot U \,\mathrm{d}(x,z) \in \mathbb{R}, && \text{(Angular Momentum)} \end{aligned}$$

satisfy the following conservation laws

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}M(\omega_t) = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}P(\omega_t) = \int_{\omega_t} f \,\mathrm{d}(x,z) + \int_{\partial\omega_t} \mathcal{T}\cdot n_{\omega_t}\,\mathrm{d}\mathscr{H}^1(x,z), \\ &\frac{\mathrm{d}}{\mathrm{d}t}L(\omega_t) = \int_{\omega_t} (-z,x)\cdot f \,\mathrm{d}(x,z) + \int_{\partial\omega_t} (-z,x)\cdot\mathcal{T}\cdot n_{\omega_t}\,\mathrm{d}\mathscr{H}^1(x,z). \end{aligned}$$

Here, ω_t denotes the image of an arbitrary reference set $\omega_0 \subset \Omega_0$ under Φ_t^U and n_{ω_t} is its outer unit normal. The new objects we have introduced are $f = f(t,x,z) \in \mathbb{R}^2$, the *volume force* and $\mathcal{T} = \mathcal{T}(t,x,z) \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, the *surface stress tensor*.

We can reframe the aforementioned conservation laws as a system of partial-differential equations, namely

$$\begin{cases} \rho\,(\partial_t U + (U\cdot\nabla)\,U) = \operatorname{div}\mathcal{T} + f & \text{(eq. of motion)} \\ \partial_t\rho + \operatorname{div}[\rho U] = 0 & \text{(eq. of continuity).} \end{cases} \tag{1.2}$$

Up to this point we have not used any specific properties of fluids. To narrow down the behaviour of our material, we introduce *constitutive laws*, which pose a dependency of the forces and stresses on the other physical quantities. In the viscous incompressible Navier-Stokes case, we consider the following set of assumptions:

i) The fluid is *isotropic*, meaning we start with a constant density $\rho(0,\cdot) = \rho_0 \mathbb{1}_{\Omega_0}$.

ii) The fluid is *incompressible*, meaning

$$\operatorname{div} U = \partial_x u + \partial_z v = 0.$$

iii) There are no external forces acting on the liquid, i.e. $f = 0$. This especially excludes gravity.

iv) The vector quantity f and the tensor quantity $\mathcal{T}$ are invariant under *a change of observer*, i.e. for all $Q \in \mathrm{SO}(2) = \{Q \in \mathbb{R}^{2\times 2} : Q^{-1} = Q^T = \mathbb{1}, \det A = 1\}$ we have

$$Q \cdot f(t,x,z) = f(t, Q\cdot(x,z)^T) \quad \text{and} \quad Q\cdot\mathcal{T}(t,x,z)\cdot Q^T = \mathcal{T}(t, Q\cdot(x,z)^T).$$

v) The stress tensor $\mathcal{T}$ depends separately on the *local deformations of the material*, $\mathcal{D} = \mathrm{D}[\Phi_t^U]\circ(\Phi_t^U)^{-1}$ and the *local changes in velocity*, $\mathrm{D}U$:

$$\mathcal{T} = \mathcal{T}_{\mathrm{el}}(\mathcal{D}) + \mathcal{T}_{\mathrm{vis}}(\mathrm{D}U),$$

with $\mathcal{T}_{\mathrm{vis}}(0) = 0$. The local deformations lead to *elastic stresses* inside the liquid, i.e. the particles are resisting rearrangement with respect to their initial configuration. On the other hand, the local changes in velocity lead to *viscous* properties of the fluid, always in relation to the current state. A good way to think about the viscous forces is as friction between adjacent layers of fluid particles.

vi) In addition, the dependency of $\mathcal{T}_{\mathrm{el}}$ on $\mathcal{D}$ is assumed to be invariant under the *material symmetry group* $\mathrm{SL}(2) = \{R \in \mathbb{R}^{2\times 2} : \det R = 1\}$, i.e.

$$\mathcal{T}_{\mathrm{el}}(\mathcal{D}) = \mathcal{T}_{\mathrm{el}}(\mathcal{D}R) \quad \forall R \in \mathrm{SL}(2).$$

vii) Of key importance in our modelling is the relation between viscous stresses $\mathcal{T}_{\mathrm{vis}}$ and local changes in velocity.

The classical assumption is that fluids are *Newtonian*, that is $\mathcal{T}_{\mathrm{vis}}$ depends linearly on $\mathrm{D}U$. If this is the case, we can derive the following form for the stress tensor:

$$\mathcal{T} = -P\mathbb{1} + 2\mu\,\mathcal{E}[U]$$

where $\mathcal{E}[U] = \frac{1}{2}(\mathrm{D}U + (\mathrm{D}U)^T)$ is the symmetric part of the gradient. The new variables are the *pressure* $P = P(t,x,z) \in \mathbb{R}$ and the *viscosity coefficient* $\mu \in \mathbb{R}$.

However, the Newtonian description fails for a lot of real world applications, which leads to the notion of *non-Newtonian fluids*. A common way to describe these is to assume that there is a possibly nonlinear dependency on the

shear rate $|\mathcal{E}[U]| := \sqrt{2\,\mathrm{tr}(\mathcal{E}[U]\cdot\mathcal{E}(U))}$ instead of taking μ to be a constant. In formula, we assume

$$\mu = \mu(T_{\mathrm{char}}\,|\mathcal{E}[U]|).$$

The unit T_{char} represents the characteristic time scale on which the non-Newtonian behaviour takes into effect. Introducing it at this point is also a convenient way to keep the argument of μ dimensionless, as each component of $\mathcal{E}[U]$ is of dimension "one over time". Of interest to us are two models for the relation between μ and $|\mathcal{E}[U]|$:

1) **Ostwald–de Waele law**
Another name that is commonly used is *power law* as the linear relation for Newtonian fluids is replaced with a simple power function. For a fixed $p > 1$ we consider

$$\mu(s) = \mu_0\, s^{\frac{1}{p-1}-1} \quad \forall s > 0. \tag{1.3}$$

This model of the relation between shear and viscosity is preferable for its mathematical simplicity.

2) **Ellis law**
The viscosity term is implicitly given by

$$\frac{1}{\mu(|s|)} = \frac{1}{\mu_0}\left(1 + \left|\frac{\mu(|s|)s}{\mathcal{T}_{\mathrm{vis},1/2}}\right|^{p-2}\right) \quad \forall s \in \mathbb{R}. \tag{1.4}$$

The constant $\mathcal{T}_{\mathrm{vis},1/2}$ corresponds to the characteristic shear rate, at which the viscosity halves. We always assume $p > 2$. This model behaves very similar to the Newtonian case for small shear rates, because there the $\frac{1}{\mu_0}$ term dominates. A fluid which satisfies Ellis law is *shear thinning*. This means its viscosity decreases for faster shear rates compared to the Newtonian situation.

For every version of μ, we require that

$$s \mapsto \mu(|s|)\, s$$

is strictly increasing. This is sensible from a physical point of view, because we expect the internal forces to increase with the strength of the deformation. We will denote the inverse map by Ψ. For power-law viscosity, Ψ is given by

$$\Psi(s) = |s|^{p-2}s. \tag{1.5}$$

Computing the inverse function for Ellis law turns out to be not necessary, because we can use the implicit formula directly in (1.23).

1.1.3 Introducing boundary conditions

Keeping μ in its general form for now, we take a look at equation (1.2) subject to the assumptions in Section 1.1.2 and all their consequences. We find the following system of PDEs

$$\begin{cases} \rho\,(\partial_t u + u\,\partial_x u + v\,\partial_z u) &= -\partial_x P + 2\,\partial_x[\mu\cdot\partial_x u] + \partial_z[\mu\cdot(\partial_z u + \partial_x v)], \\ \rho\,(\partial_t v + u\,\partial_x v + v\,\partial_z v) &= -\partial_z P + 2\,\partial_z[\mu\cdot\partial_z v] + \partial_x[\mu\cdot(\partial_z u + \partial_x v)], \\ \partial_x u + \partial_z v &= 0, \end{cases} \tag{1.6}$$

where we suppress the dependency of μ on $|\mathcal{E}[U]|$ for readability.

Equation (1.6) describes how the fluid flows inside the domain $\Omega = \{(t,x,z) : 0 < z < h(t,x)\}$, but it does not explain how Ω itself evolves. To address this, we need to couple the variable h with the other unknowns u, v, P. We impose the following set of boundary conditions at the *upper* boundary $\{(t,x,z) : z = h(t,x)\}$:

$$\begin{cases} \partial_t h + u\,\partial_x h = v, \\ \mathcal{T}\,\mathrm{n} = \sigma\,\kappa\,\mathrm{n} \end{cases} \tag{1.7}$$

with

$$\mathrm{n} = \frac{1}{(1+|\partial_x h|^2)^{1/2}}\begin{pmatrix} -\partial_x h \\ 1 \end{pmatrix}, \qquad \kappa = \frac{\partial_x^2 h}{(1+|\partial_x h|^2)^{3/2}},$$

being the *outer unit normal* and the *mean curvature* of the surface $\{(x,z) : z = h(t,x)\}$ for a given t. The very first condition in (1.7) is called the *kinematic boundary condition*, which in essence forces particles that start on the boundary to stay on the boundary. Indeed, if we look at a pair (x_0, z_0), such that $h(0,x_0) = z_0$, we require

$$h(t,x_t) = z_t,$$

for all times t,where $(x_t, z_t) := \Phi_t^U(x_0, z_0)$ (Recall the definition of the flow Φ^U in (1.1)). Differentiation in time yields the kinematic equation.

The second assumption relates the stresses in the fluid to the curvature of the boundary. Surface tension encourages the interface to flatten out. We say that the fluid is *capillary driven*.

So far we have discussed the modelling for the interior of the fluid and for the free upper-boundary. What is left is the behaviour at the solid interface, which has been an active discussion throughout the development of thin film theory. The central question is: *How does a liquid slide over a surface?*

The classical option is to assume a *no-slip* condition, meaning

$$u = 0 \quad \text{at } z = 0.$$

As we will compute in Section 1.1.5, in the Newtonian rheology this leads to the thin-film equation

$$\partial_t h + \partial_x[h^3\,\partial_x^3 h] = 0, \tag{1.8}$$

which unfortunately features the so-called *no-slip paradox*, where the energy dissipation at the contact point becomes infinite and thus physically not sensible. This behaviour was discovered in [HS71] and [DD74] and we refer to the source for details.

A resolution to the problem comes via a slippage assumption due to C. Navier in [Nav23], now known as the *Navier-slip* condition:

$$u = \lambda \, \partial_z u \quad \text{at } z = 0.$$

The parameter $\lambda \geq 0$ is called the *slip length*. In this model, the horizontal velocity at the solid is linearly dependent to the horizontal shear, i.e. how much the particle layers move against each other. The resulting thin-film equation features a power of two on h in the divergence part and thereby circumvents the no-slip paradox.

Based on the Navier-slip assumptions one can introduce more general types of slippage conditions, for example featuring a non-local dependency on the film height

$$u = \Lambda(h) \, \partial_z u \quad \text{at } z = 0.$$

In the end we will follow the choice in [GGO16], which reads

$$\Lambda(h) = \lambda^{3-m} \, h^{m-2}, \tag{1.9}$$

for a given exponent $m \in [0, 3)$.

In the regime $m \in [0, 2)$ the power on h is negative, meaning the slippage increases as the fluid height tends to zero. On the remaining interval $m \in (2, 3)$, the opposite happens.

Next to the horizontal slip at the interface, we also need to fix a condition for the vertical velocity of our fluid. We will impose

$$v = 0 \quad \text{at } z = 0,$$

which we interpret as "particles at the solid stay at the solid".

1.1.4 Dimensionless equation and lubrication approximation

Our goal is to derive an equation where only h, x and t remain as variables and we get there by applying the so-called lubrication approximation. Roughly speaking, we let the aspect ratio between the height and the width of the fluid tend to zero and identify the terms of leading order. The idea dates back to Reynolds in [Rey86], who used this process to derive an equation for the pressure in a lubricant between two horizontal plates.

The following computations are purely formal, although some results exist that turn the heuristics into rigorous mathematics, such as in [GO03].

To perform the rescaling we need to pick scales in space and time. Let H be a *characteristic height* of the fluid and L a *characteristic length* and call T the *characteristic time scale* of our system. Common choices are the average film

height as H and the width as L. We are interested in the ratio $\varepsilon = \frac{H}{L}$ and how the system changes as we let $\varepsilon \to 0$. Starting with the space-time variables, we define

$$\bar{x} = \frac{x}{L}, \qquad \bar{z} = \frac{z}{H}, \qquad \bar{t} = \varepsilon^3 \frac{t}{T}.$$

For the components of our velocity field U, the dimensionless unknowns are

$$\bar{u} = \frac{T}{L\varepsilon^3} u, \qquad \bar{v} = \frac{T}{H\varepsilon^3} v.$$

Turning to the right-hand side of (1.6), we fix μ_0 as the *characteristic viscosity* of the fluid. Using this, we can rewrite the pressure as

$$\bar{P} = \frac{T}{\mu_0 \varepsilon} P.$$

The viscosity term itself is a bit more tricky, since by our non-Newtonian modelling it depends on the shear rate. Introducing $\tau = \frac{T_{\text{char}}}{T}\varepsilon^3$, we write

$$T_{\text{char}}\, \mathcal{E}[U] = \tau \begin{pmatrix} \varepsilon\, \partial_{\bar{x}} \bar{u} & \frac{1}{2}(\partial_{\bar{z}} \bar{u} + \varepsilon^2 \partial_{\bar{x}} \bar{v}) \\ \frac{1}{2}(\partial_{\bar{z}} \bar{u} + \varepsilon^2 \partial_{\bar{x}} \bar{v}) & \varepsilon\, \partial_{\bar{z}} \bar{v} \end{pmatrix} =: \tau\, \bar{\mathcal{E}}(\bar{U}).$$

Note, that if we let $\varepsilon \to 0$, we formally satisfy $\left|\bar{\mathcal{E}}(\bar{U})\right| \to |\partial_{\bar{z}} \bar{u}|$. This is enough to define the dimensionless viscosity function:

$$\bar{\mu}(\tau \left|\bar{\mathcal{E}}(\bar{U})\right|) := \frac{1}{\mu_0} \mu(T_{\text{char}} \left|\mathcal{E}[U]\right|).$$

Finally, we address the quantities at the upper and lower boundary,

$$\bar{h} = \frac{1}{H} h, \qquad \bar{\sigma} = \frac{T}{L\mu_0} \sigma, \qquad \bar{\lambda} = \frac{\lambda}{H}.$$

We can now rewrite (1.6) in terms of the barred variables. We compute

$$\partial_t u + u\, \partial_x u + v\, \partial_z u = \varepsilon^6 \frac{L}{T^2} \left(\partial_{\bar{t}} \bar{u} + \bar{u}\, \partial_{\bar{x}} \bar{u} + \bar{v}\, \partial_{\bar{z}} \bar{u}\right) \tag{1.10}$$

$$\partial_t v + u\, \partial_x v + v\, \partial_z v = \varepsilon^7 \frac{L}{T^2} \left(\partial_{\bar{t}} \bar{v} + \bar{u}\, \partial_{\bar{x}} \bar{v} + \bar{v}\, \partial_{\bar{z}} \bar{v}\right). \tag{1.11}$$

On the right-hand side,

$$\begin{aligned} &-\partial_x P + 2\,\partial_x[\mu\, \partial_x u] + \partial_z[\mu\,(\partial_z u + \partial_x v)] \\ &\qquad = \frac{\mu_0}{LT}\left(-\varepsilon \partial_{\bar{x}} \bar{P} + 2\varepsilon^3\, \partial_{\bar{x}}[\bar{\mu}\, \partial_{\bar{x}} \bar{u}] + \varepsilon\, \partial_{\bar{z}}[\bar{\mu}\, \partial_{\bar{z}} \bar{u}] + \varepsilon^3\, \partial_{\bar{z}}[\bar{\mu}\, \partial_{\bar{x}} \bar{v}]\right) \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} &-\partial_z P + 2\,\partial_z[\mu\, \partial_z v] + \partial_x[\mu\,(\partial_z u + \partial_x v)] \\ &\qquad = \frac{\mu_0}{LT}\left(-\partial_{\bar{z}} \bar{P} + 2\,\varepsilon^2 \partial_{\bar{z}}[\bar{\mu}\, \partial_{\bar{z}} \bar{v}] + \varepsilon^2 \partial_{\bar{x}}[\bar{\mu}\, \partial_{\bar{z}} \bar{u}] + \varepsilon^4\, \partial_{\bar{x}}[\bar{\mu}\, \partial_{\bar{x}} \bar{v}]\right). \end{aligned} \tag{1.13}$$

Once we multiply the equality between (1.10) and (1.12) with $\frac{LT}{\varepsilon\mu_0}$ and the one between (1.11) and (1.13) with $\frac{LT}{\mu_0}$, we can formally send $\varepsilon \to 0$. The system that remains is

$$\begin{cases} \partial_{\bar{x}} \bar{P} = \partial_{\bar{z}} [\bar{\mu}(\tau\, |\partial_{\bar{z}} \bar{u}|)\, \partial_{\bar{z}} \bar{u}], \\ \partial_{\bar{z}} P = 0, \\ \partial_{\bar{x}} \bar{u} + \partial_{\bar{z}} \bar{v} = 0 \end{cases}$$

By moving all structural constants to the left-hand side, we have grouped them into the *Reynolds number*, a classical parameter to describe the properties of our fluid:

$$\mathrm{Re} := \frac{\rho\, \varepsilon^3\, L^2}{\mu_0\, T}.$$

The dependency on ε^3 shows that the lubrication approximation takes place in the regime of small Reynolds number, meaning we consider laminar flows, for which the fluid layers do not mix.

Next, we investigate how the boundary conditions scale. We work on the rescaled domain $\bar{\Omega} = \{(\bar{t}, \bar{x}, \bar{z}) : 0 < \bar{z} < \bar{h}(\bar{t}, \bar{x})\}$ and its boundary in the barred variables. The kinematic boundary condition reads

$$0 = \partial_t h + u\, \partial_x h - v = \varepsilon^3\, \frac{H}{T} (\partial_{\bar{t}} \bar{h} + \bar{u}\, \partial_{\bar{x}} \bar{h} - \bar{v})$$

and is therefore present whilst taking the limit.

For the capillary equations left-hand side, we compute

$$\begin{aligned} \mathcal{T}\mathrm{n} &= \frac{1}{(1 + |\partial_x h|^2)^{1/2}} \begin{pmatrix} -\partial_x h\, (2\, \mu\, \partial_x u - P) + \mu\, (\partial_z u + \partial_x v) \\ -\partial_x h\, \mu\, (\partial_z u + \partial_x v) + 2\mu \partial_z v - P \end{pmatrix} \\ &= \frac{\mu_0}{T} \frac{1}{(1 + \varepsilon^2 \left|\partial_{\bar{x}} \bar{h}\right|^2)^{1/2}} \begin{pmatrix} -\varepsilon\, \partial_{\bar{x}} \bar{h}\, (\varepsilon^3\, 2\, \bar{\mu}\, \partial_{\bar{x}} \bar{u} - \varepsilon \bar{P}) + \bar{\mu}\, (\varepsilon^2 \partial_{\bar{z}} \bar{u} + \varepsilon^4 \partial_{\bar{x}} \bar{v}) \\ -\varepsilon\, \partial_{\bar{x}} \bar{h}\, \bar{\mu}\, (\varepsilon^2\, \partial_{\bar{z}} \bar{u} + \varepsilon^4\, \partial_{\bar{x}} \bar{v}) + \varepsilon^3\, 2 \bar{\mu} \partial_{\bar{z}} \bar{v} - \varepsilon \bar{P} \end{pmatrix}, \end{aligned}$$

and for the right-hand side

$$\sigma\, \kappa\, \mathrm{n} = \frac{\sigma\, \partial_x^2 h}{(1 + |\partial_x h|^2)^2} \begin{pmatrix} -\partial_x h \\ 1 \end{pmatrix} = \varepsilon \frac{\mu_0}{T} \frac{\bar{\sigma}\, \partial_{\bar{x}}^2 \bar{h}}{(1 + \varepsilon^2 \left|\partial_{\bar{x}} \bar{h}\right|^2)^2} \begin{pmatrix} -\varepsilon \partial_{\bar{x}} \bar{h} \\ 1 \end{pmatrix}.$$

We extract the lowest-order terms to obtain the limit equations

$$\begin{aligned} \partial_{\bar{z}} \bar{u} &= 0 \\ \bar{P} &= -\bar{\sigma}\, \partial_{\bar{x}}^2 \bar{h} \end{aligned}$$

on the boundary $\{(\bar{t}, \bar{x}, \bar{z}) : \bar{z} = \bar{h}(\bar{t}, \bar{x})\}$.

At the bottom boundary the rescaling poses no difficulty, once we replace $\lambda = H\, \bar{\lambda}$ in (1.9). This way we keep everything dimensionless and all slippage conditions carry naturally beyond the lubrication approximation process.

1.1.5 Reduction to a single evolution equation

Let us recollect the model so far. We will treat everything in dimensionless variables and take the liberty to drop the bar on each symbol. After lubrication approximation, we consider the system

$$\begin{cases} \partial_x P = \partial_z[\mu(\tau\,|\partial_z u|)\,\partial_z u], & \text{(1.14a)} \\ \partial_z P = 0, & \text{(1.14b)} \\ \partial_x u + \partial_z v = 0 & \text{(1.14c)} \end{cases}$$

in $\Omega = \{(t,x,z) : 0 < z < h(t,x)\}$, subject to the conditions on the upper boundary

$$\begin{cases} \partial_t h + u\,\partial_x h = v, & \text{(1.15a)} \\ P = -\sigma\,\partial_x^2 h, & \text{(1.15b)} \\ \partial_z u = 0 & \text{(1.15c)} \end{cases}$$

at $\partial\Omega \cap \{z > 0\}$ and the slippage assumption at the lower boundary

$$\begin{cases} u = \Lambda(h)\,\partial_z u, & \text{(1.16a)} \\ v = 0 & \text{(1.16b)} \end{cases}$$

at $\partial\Omega \cap \{z = 0\}$. We will now present the first result of this thesis: a thin-film equation for non-Newtonian fluids with general slippage condition.

Theorem 1. *The system (1.14) subject to (1.15) and (1.16) is equivalently described by the evolution of the free boundary h, via the equation*

$$\partial_t h + \tfrac{1}{\tau}\partial_x\left[\Lambda(h)h\,\Psi(\tau\sigma\,h\,\partial_x^3 h) + \int_0^h \zeta\,\Psi(\tau\sigma\,\zeta\,\partial_x^3 h)\,\mathrm{d}\zeta\right] = 0. \tag{1.17}$$

Note that we consider ∂_x as an operator, so it also falls onto the h on the integral symbol.

Proof. We need to find expressions for our unknowns u, v and P entirely in terms of h. The simplest of these is the pressure. Equation (1.14b) implies that the pressure is constant along vertical lines, so it can be derived from the capillary equation at the upper boundary (1.15b):

$$P(t,x,z) = -\sigma\,\partial_x^2 h(t,x). \tag{1.18}$$

Up next is the horizontal component u of the velocity field. We differentiate (1.18) in x, plug the result in (1.14a) and then integrate vertically from the upper boundary to a point (t,x,z) in the bulk, to find

$$\mu(\tau\,|\partial_z u|)\,\partial_z u = \sigma\,(h - z)\,\partial_x^3 h. \tag{1.19}$$

Here we used (1.15c), to get rid of the evaluation of $\partial_z u$ at the boundary $z = h$. At this point we make use of assumption Equation (1.5) that there exists an inverse function Ψ to the map $s \mapsto \mu(|s|)s$. We may solve (1.19) for $\partial_z u$:

$$\partial_z u = \tfrac{1}{\tau}\Psi(\tau\sigma\,(h - z)\,\partial_x^3 h). \tag{1.20}$$

Integrating vertically again, but this time from the bottom up to (t, x, z), we find an expression for u, using the slippage condition (1.16a) with (1.20) inserted:

$$u = \tfrac{1}{\tau}\left(\Lambda(h)\,\Psi(\tau\sigma\, h\,\partial_x^3 h) + \int_0^z \Psi(\tau\sigma\,(h-\zeta)\,\partial_x^3 h)\,\mathrm{d}\zeta\right). \tag{1.21}$$

The vertical velocity v, can be obtained using the incompressibility condition (1.14c) together with the lower boundary data (1.16b). In formula

$$\begin{aligned} v &= v|_{z=0} + \int_0^z \partial_z v\,\mathrm{d}\zeta \\ &= -\int_0^z \partial_x u\,\mathrm{d}\zeta \\ &= -\partial_x\left[\int_0^z \tfrac{1}{\tau}\left(\Lambda(h)\,\Psi(\tau\sigma\, h\,\partial_x^3 h) + \int_0^\zeta \Psi(\tau\sigma\,(h-\xi)\,\partial_x^3 h)\,\mathrm{d}\xi\right)\mathrm{d}\zeta\right] \\ &= -\tfrac{1}{\tau}\partial_x\left[\Lambda(h) z\,\Psi(\tau\sigma\, h\,\partial_x^3 h) + \int_0^z (z-\zeta)\,\Psi(\tau\sigma\,(h-\zeta)\,\partial_x^3 h)\,\mathrm{d}\zeta\right]. \end{aligned}$$

The double integral was resolved by using Fubini.

The final thing we need to take care of is the evolution equation for h itself. Here, we rewrite the kinematic equation (1.15a) as

$$\partial_t h + \partial_x\left[\int_0^h u\,\mathrm{d}\zeta\right] = 0,$$

which is possible due to (1.16b). A similar calculation as for the vertical velocity yields the desired result

$$\partial_t h + \tfrac{1}{\tau}\partial_x\left[\Lambda(h) h\,\Psi(\tau\sigma\, h\,\partial_x^3 h) + \int_0^h \zeta\,\Psi(\tau\sigma\,\zeta\,\partial_x^3 h)\,\mathrm{d}\zeta\right] = 0.$$

This concludes the derivation. □

1.1.6 Specialisation to Ostwald–de Waele and Ellis fluids

Let us adapt the general equation in Theorem 1 to the explicit examples for the viscosity function μ, the power law in (1.3) and Ellis law in (1.4).

Starting with power law, we stick to the slippage condition in (1.9) and perform a rescaling in the time variable to get rid of the constants τ and σ. We obtain the following equation

$$\partial_t h + \partial_x\left[\left(\gamma_{\mathrm{P}}\, h^{m+p-2} + h^{p+1}\right)\left|\partial_x^3 h\right|^{p-2}\partial_x^3 h\right] = 0, \tag{1.22}$$

with $\gamma_{\mathrm{P}} = (1+\frac{1}{p})\lambda^{3-m}$.

In the literature this equation is simplified even further, by assuming that the slip-length λ is either zero or proportionally large. The first case results in the thin-film equation

$$\partial_t h + \partial_x\left[h^{p+1}\left|\partial_x^3 h\right|^{p-2}\partial_x^3 h\right] = 0,$$

which notably becomes (1.8) if we are in Newtonian rheology, $p = 2$. The second case gets mathematically realised by performing an appropriate rescaling in time and then grouping by the highest-order terms of λ, just like we did for the lubrication approximation in Section 1.1.4. Hence, formally this situation leads to equations of the type

$$\partial_t h + \partial_x \left[h^n \left| \partial_x^3 h \right|^{p-2} \partial_x^3 h \right] = 0,$$

where $n = m + p - 2 \in [0, p+1)$.

The Navier-slip case, $m = 2$ and $p = 2$, reads

$$\partial_t h + \partial_x \left[h^2 \, \partial_x^3 h \right] = 0.$$

For Ellis law, it is easier to employ the implicit formula for μ from (1.4). If we jump back into (1.19) and rewrite it using (1.4), we obtain

$$\partial_z u = \sigma \, (h - z) \, \partial_x^3 h \left(1 + \left| \frac{\tau \sigma \, (h - z) \, \partial_x^3 h}{\mathcal{T}_{\mathrm{vis},1/2}} \right|^{p-2} \right). \tag{1.23}$$

Continuing the proof of Theorem 1, with this explicit expression, we arrive at the following thin-film equation

$$\partial_t h + \partial_x \left[(\lambda^{3-m} h^m + \tfrac{1}{3} h^3) \, \partial_x^3 h \right] \tag{1.24}$$

$$+ \tilde{\gamma_{\mathrm{E}}} \, \partial_x \left[(\lambda^{3-m} \, h^{m+p-2} + \tfrac{1}{p} h^3) \left| \partial_x^3 h \right|^{p-2} \partial_x^3 h \right] = 0, \tag{1.25}$$

with $\tilde{\gamma_{\mathrm{E}}} = \left| \frac{\tau \sigma}{\mathcal{T}_{\mathrm{vis},1/2}} \right|^{p-2}$. Once more, we either set $\lambda = 0$ or rescale properly to remove all lower-order terms, to obtain an equation of the form

$$\partial_t h + \partial_x \left[h^n \, (\gamma_{\mathrm{E}} + \left| h \, \partial_x^3 h \right|^{p-2}) \, \partial_x^3 h \right] = 0,$$

with $n \in [0, 3]$ and $\gamma_{\mathrm{E}} > 0$.