§1 Introduction

Montag, 21. November 2022 13:50

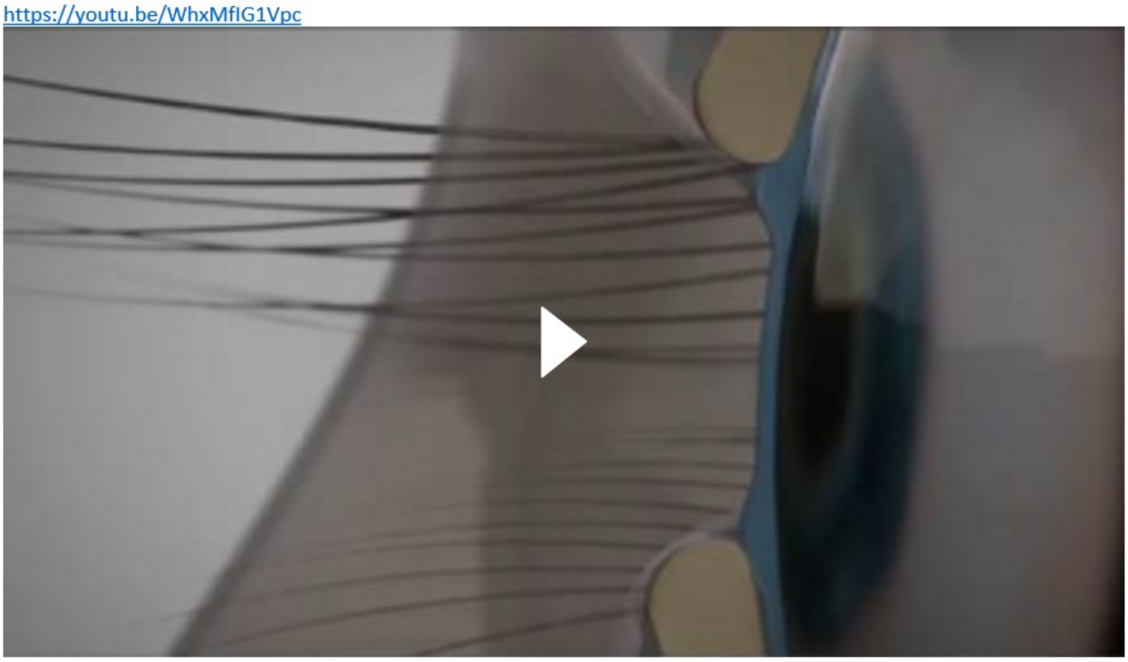
§1.1 what are thin-films?

* a body of fluid, shallow in one dimension

Examples

- Tear film in the eye
  * surface tension & gravity are at play

YT: AllThingsEyes
https://youtu.be/WhxMfIG1Vpc



- Lava flows:
  - on large scales: a few meters are thin
  - viscosity is key factor

YT: Olivier Grunewald [My drone above the incredible Icelandic geyser of lava]

- Analogue Film production
  - Liquid-Liquid interactions
  - don't want turbulence! want laminar flow.
  - liquid-solid interaction

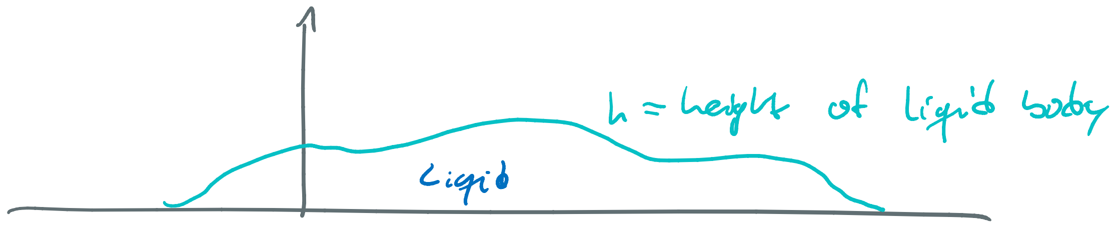
YT: SmarterEveryDay
https://youtu.be/cAAJUHwh9F4

§1.2 what are thin-film equations?

- Main idea: By assuming $\frac{\text{height}}{} \ll 1$

* Main idea: By assuming $\frac{\text{height}}{\text{Length}} \ll 1$

we can reduce the question: "How does the liquid evolve?" to a single equation for the fluid's height.



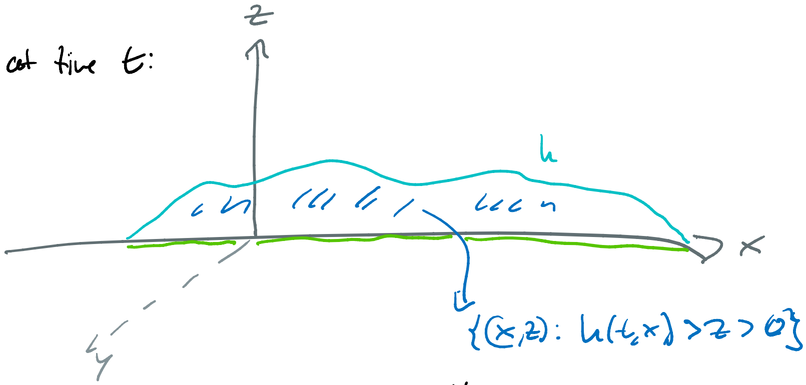
* Common in literature: only in one horizontal direction.

Goal derive this equation.

§2 Derivation of a thin-film equation

Mittwoch, 23. November 2022 10:01

at time t:



Need physics at three "zones"

1) Interior
2) Upper boundary
3) Lower boundary

Zone 1 Interior

We use incompressible Navier-Stokes system

$u = u(t,x,z) \mathrel{\hat=}$ horizontal velocity

$v = v(t,x,z) \mathrel{\hat=}$ vertical velocity

$\pi = \pi(t,x,z) \mathrel{\hat=}$ pressure

$\rho = \text{const} \mathrel{\hat=}$ density

$\mu = \text{const} \mathrel{\hat=}$ viscosity

$m\,a \quad = \quad F$ — pressure, viscosity

$$\Rightarrow \begin{cases} \rho(\partial_t u + u\,\partial_x u + v\,\partial_z u) = -\partial_x \pi + \mu(\partial_x^2 + \partial_z^2) u \\ \rho(\partial_t v + u\,\partial_x v + v\,\partial_z v) = -\partial_z \pi + \mu(\partial_x^2 + \partial_z^2) v \\ \partial_x u + \partial_z v = 0 \end{cases}$$

inside $\{(t,x,z):\ h(t,x) > z > 0\}$

Zone 2 Upper boundary

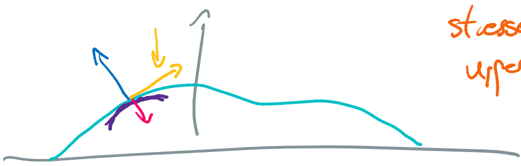
- Need to couple h to the system.

  Kinematic equation: $\partial_t h + u\,\partial_x h = v$

- Capillary equation:

surface tension

$$\underbrace{\left(-\pi \mathbb{1} + \frac{\mu}{2}\left(D\binom{u}{v} + D\binom{u}{v}^T\right)\right)}_{\text{stresses at the upper boundary}} \underbrace{\binom{-\partial_x h}{1}}_{\text{normal on the graph of } h} = \underbrace{\frac{\sigma\, \partial_x^2 h}{(1+(\partial_x h)^2)^{3/2}}}_{\text{curvature of the graph of } h} \binom{-\partial_x h}{1}$$



Zone 3 Lower boundary

Q: How do liquid particles interact with a solid?

- No-slip assumption: $u = 0$ at $\{z = 0\}$

This has the no-slip paradox!

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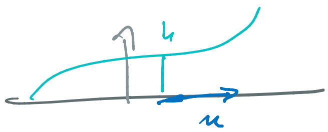
• Navier-slip assumption: $u = \lambda\, \partial_z u$ at $\{z=0\}$



• More general assumption exist. $u = \lambda^{3-n} h^{n-2} \partial_z u$ $\quad n=2$, $n\in(1,3)$

Also $v=0$!

§2.2 Lubrication approximation

- Navier-Stokes system
- Capillary equation
- Kinematic equation
- No-slip

Look at this non-dimensionalised

• $x = L\bar{x}$, $z = H\bar{z}$, $\varepsilon = \frac{H}{L}$, $t = \frac{T}{\varepsilon^3}\bar{t}$

• $u = \frac{L}{T}\varepsilon^3\bar{u}$, $v = \frac{H}{T}\varepsilon^3\bar{v}$, $\pi = \frac{\mu}{T}\varepsilon\bar{\pi}$

$\Rightarrow$ $\boxed{\text{pressure} \ll \text{viscosity} \sim \text{surface tension}}$

• $h = H\bar{h}$, $\sigma = \frac{\mu L}{T}\bar{\sigma}$, $Re = \frac{\rho\,\varepsilon^3 L^2}{\mu\, T}$

$$\Rightarrow \underline{\varepsilon^2} Re\left(\partial_{\bar t}\bar{u} + \bar{u}\,\partial_{\bar x}\bar{u} + \bar{v}\,\partial_{\bar z}\bar{v}\right) = \boxed{-\partial_{\bar x}\bar{\pi}} + \Big(\underbrace{\varepsilon^2\partial_{\bar x}^2\bar{u}}_{0} + \boxed{\partial_{\bar z}^2\bar{u}}\Big)$$

$$\Big\downarrow\ \varepsilon\searrow 0$$

$$(1)\quad \begin{cases} \partial_{\bar x}\bar{\pi} = \partial_{\bar z}^2\bar{u} \\ \partial_{\bar z}\bar{\pi} = 0 \\ \partial_{\bar x}\bar{u} + \partial_{\bar z}\bar{v} = 0 \end{cases}$$

• Same goes for kinematic equation $\rightsquigarrow (\tilde{2})\ \partial_{\bar t}\bar{h} + \bar{u}\,\partial_{\bar x}\bar{h} = \bar{v}$.

• Capillary eq: $\xrightarrow[\varepsilon\searrow 0]{}$ $(2)\ \begin{cases} \partial_{\bar z}\bar{u} = 0 \\ \bar{\pi} = -\bar{\sigma}\,\partial_{\bar x}^2\bar{h} \end{cases}$ at $\{\bar{h} = \bar{z}\}$

• $\bar{u} = 0$, $\bar{v} = 0$ $\quad (3)$

Note: $Re = \frac{\rho\,\varepsilon^2 L^2}{\mu\, T} \ll 1$. Small vertical velocities!
$\Rightarrow$ film needs to stay small.

Goal compress this into a single equation for h.

Idea express u, v, π in terms of h $\Rightarrow$ Kinematic equation would compress ...

Idea: express u, v, Π in terms of h $\Rightarrow$ Kinematic equation

Ad Π No more barred variables.

$$\Pi = -\sigma \partial_x^2 h \quad \text{at } z = h \qquad \Rightarrow \quad \Pi = -\sigma \partial_x^2 h \quad \forall (x,z):\ h(t,x) > z > 0$$

$$\partial_z \Pi = 0$$

Ad u

$$\partial_z^2 u = \partial_x \Pi = -\sigma \partial_x^3 h \quad \text{in } \{h > z > 0\}$$

$$\partial_z u = 0 \quad \text{at } z = h$$

$$u = 0 \quad \text{at } z = 0$$

$$\partial_z u = \underbrace{\partial_z u|_{z=h}}_{=0} - \int_z^h \partial_z^2 u \, d\xi = \sigma \int_z^h \partial_x^3 h \, d\xi = \sigma (h - z) \partial_x^3 h$$

$$u = u|_{z=0} + \int_0^z \partial_z u \, d\xi = \sigma \int_0^z (h - \xi) \partial_x^3 h \, d\xi$$

$$= \sigma \partial_x^3 h \left(zh - \tfrac{1}{2} z^2 \right)$$

Ad v similar calculation using $\partial_x u = -\partial_z v$ & $v|_{z=0} = 0$

Ad h insert u, v into $\partial_t h + u \partial_x h - v = 0$

$$0 = \partial_t h + u \partial_x h - v$$

$$= \partial_t h + \partial_x \left[\int_0^h u \, dz \right] \qquad \partial_x \left[\int_0^h u \, dz \right] = \partial_x h \cdot u + \int_0^h \partial_x u \, dz$$

$$= \partial_x h \cdot u - \int_0^h \partial_z v \, dz$$

$$= \partial_x h \cdot u - v$$

$$= \partial_t h + \partial_x \left[\int_0^h \sigma \partial_x^3 h \left(zh - \tfrac{1}{2} z^2 \right) dz \right]$$

$$= \partial_t h + \partial_x \left[\sigma \partial_x^3 h \left(\tfrac{1}{2} h^3 - \tfrac{1}{6} h^3 \right) \right]$$

$$= \partial_t h + \frac{\sigma}{3} \partial_x \left[h^3 \partial_x^3 h \right]$$

coeff of highest order derivative

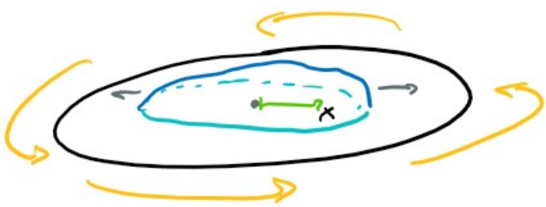
- Nonl 4-th order
- degenerate, since the coeff of the highest order can vanish.

§3 Examples

Mittwoch, 23. November 2022 09:55

Examples

- Spin-coating (↗ Gnann et. al. 2017)
  - * Fluid on a rotating plate
  - * centrifugal forces act with surface tension



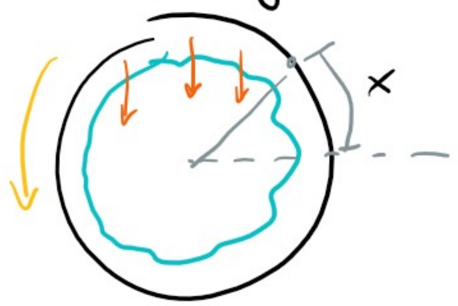
$$\Longrightarrow \partial_t h + \underbrace{\frac{\partial_x\left[x^2 (h^2+h^3)\right]}{x}}_{\text{centrifugal}} + \underbrace{\frac{\partial_x\left[x\,(h^2+h^3)\,\partial_x\left[\frac{\partial_x[x\,\partial_x h]}{x}\right]\right]}{x}}_{\text{surface tension}} = 0$$

∘ 4-th order equation.

• Rimming flows

↗ Moffat 1977

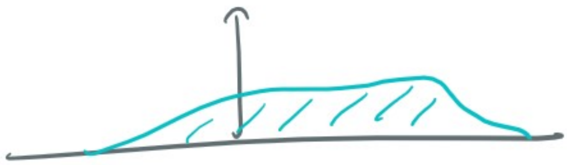
↗ Pukhnachev 1977



- gravity acts
- outside rotates w/ constant speed.

$$\Rightarrow \partial_t h + \partial_x \left[h - b h^3 \cos(x) + a h^3 \left(\partial_x h + \partial_x^3 h \right) \right] = 0$$

• Horizontal plate w/o gravity



$$\Rightarrow \partial_t h + \partial_x \left[h^3 \partial_x^3 h \right] = 0.$$