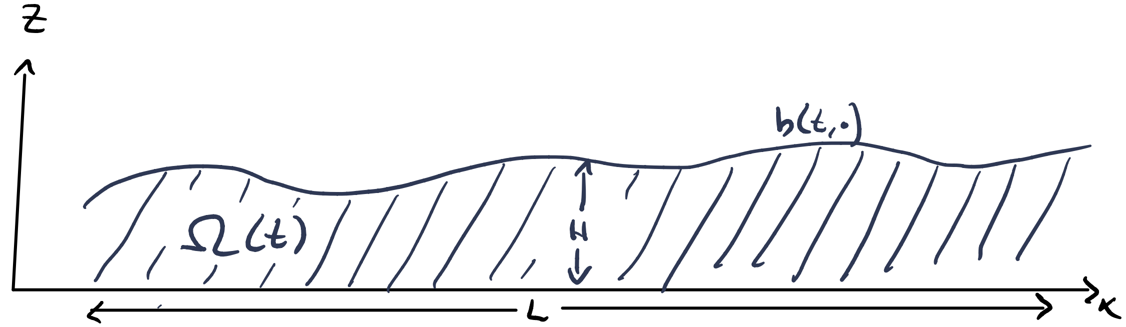
Short-time existence and standard parabolic theory

1. Ouverture / Reminder

- incompressible,
  viscous,
  Newtonian fluid
- homogeneous in y-direction



Lubrication approximation: asymptotic model for vanishing aspect ratio $\varepsilon = \frac{H}{L} \longrightarrow 0$

start from Navier-Stokes system $\vec{u} = (u, v)$

$$
\begin{cases}
Re\left(\partial_t \vec{u} + (\vec{u}\cdot\nabla)\vec{u}\right) = -\nabla\Pi + \Delta\vec{u} & \text{in } \Omega(t) \\
\operatorname{div} \vec{u} = 0 & \\
u = 0 & \text{on } z = 0 \quad \text{slip condition} \\
\partial_t h + u\,\partial_x h = 0 & \text{on } z = h \quad \text{kinematic b.c.} \\
\Sigma(\vec{u}, \Pi)n = \sigma \kappa n & \text{on } z = h \quad \text{stress-balance}
\end{cases}
$$

asymptotic expansion in $\varepsilon = \frac{H}{L}$ and $\varepsilon \to 0$

$$\partial_t h + \partial_x \left(\frac{\sigma}{3} h^3 \partial_x^3 h \right) = 0 \quad \text{on } \{h > 0\}$$

More generally, we get

$$\partial_t h + \partial_x (h^n \partial_x^3 h) = 0 \quad \text{in } \{h > 0\} \qquad n \geq 1$$

Features

- fourth-order equation
  $\longrightarrow$ no comparison principle (cf. $\partial_t u + \partial_x^4 u = 0$)
  $\hookrightarrow$ solutions that are initially positive might not stay positive $\longrightarrow$ see Talk 04
- quasilinear equation

$$\partial_t h + h^n \partial_x^4 h = - (\partial_x h^n) \partial_x^3 h$$

- degenerate - parabolic
  $\longrightarrow$ parabolicity ceases as $h \to 0$

Remark: Porous medium equation

- Adding $-g\,\vec{e}_z$ in Navier-Stokes system and assuming $g \gg \sigma$

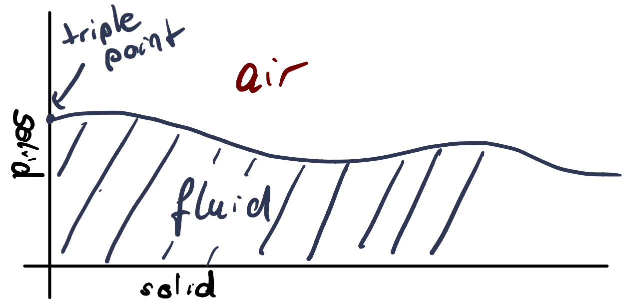
$$\rightsquigarrow \quad \partial_t h - \partial_x(h^n \partial_x h) = 0$$

Thin-film equation on bounded domains $\Omega = (a,b) \subset \mathbb{R}$

$\rightarrow$ need two boundary conditions for closed system

1. Contact angle:

   surface tension equilibrium fixes contact angle, e.g.



$$\boxed{\partial_x h = 0 \quad \text{on } \partial\Omega}$$

2. No-flux condition through boundary:

$$0 = \frac{d}{dt}\int_\Omega h(t,x)\,dx = \int_\Omega \partial_t h(t,x)\,dx = -\int_\Omega \partial_x(h^n \partial_x^3 h)\,dx = -\int_{\partial\Omega} h^n \partial_x^3 h \, d\mathcal{H}^0$$

$$\boxed{h^n \partial_x^3 h = 0 \quad \text{on } \partial\Omega}$$

Summary: we want to study

$$\begin{cases} \partial_t h + \partial_x (h^n \partial_x^3 h) = 0, & t>0,\ x\in\Omega \\ \partial_x h = \partial_x^3 h = 0, & t>0,\ x\in\partial\Omega \\ h(0,\cdot) = h_0 > 0, & x\in\Omega \end{cases}$$

2. "Standard parabolic theory": analytic semigroups in a nutshell

Goal: short-time existence + maximal regularity

Method: write equation as Cauchy problem

$$\begin{cases} \partial_t h + A[h]h = F(h) & \text{in } \Omega \\ B(h) = 0 & \text{on } \partial\Omega \\ u(0) = u_0 \end{cases}$$

where $A[g]h = g^n \partial_x^4 h$, $\quad B(h) = \begin{pmatrix} \partial_x h \\ \partial_x^3 h \end{pmatrix}$

$F(h) := -\partial_x(g^n)\partial_x^3 h$,

and use semigroup theory + fixed-point arguments

Semigroups for bounded operators

Cauchy problem: A bounded operator on X

$$\begin{cases} \partial_t u + Au = 0 \\ u(0) = u_0 \end{cases}$$

has solution $u(t) = T(t)u_0 = e^{-tA}u_0$

Note:

1.) $T(0) = \mathrm{Id}$

2.) $T(t+s) = T(t)T(s) \quad \forall s,t \geq 0$

3.) $\lim_{t \to 0} T(t)x = x \quad \forall x \in X$

$(T(t))_{t \geq 0}$ is a C_0-semigroup

4.) $\lim_{t \searrow 0} \frac{T(t)x - x}{t} = -Ax$

$-A$ is the generator of the semigroup

Recall: $e^{-tA} = \frac{1}{2\pi i} \int_{\partial B_R(0)} e^{t\lambda} (\lambda + A)^{-1} d\lambda$

Reminder: $e^{-tx} = \frac{1}{2\pi i} \int_{\partial B_R(0)} \frac{e^{t\lambda}}{\lambda + x} d\lambda$

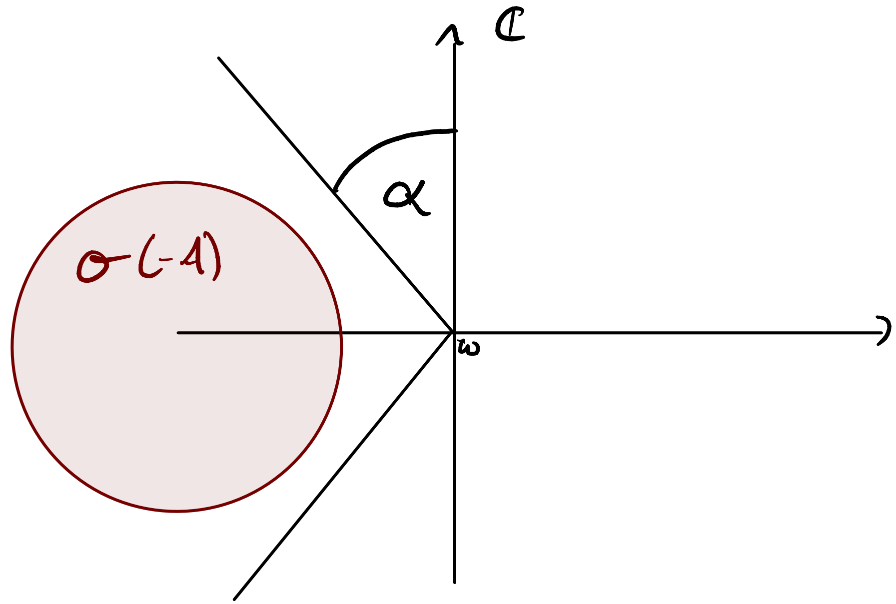
Analytic semigroups

Fix unbounded operator $A: D(A) \subset X \to X$, $\overline{D(A)} = X$ Banach space,

e.g. $A = g^n \partial_x^4$ for $g \in C^\infty(\overline{\Omega})$, $g > 0$, $X = L^2(\Omega)$; $D(A) \sim H^4(\Omega)$ + b.c.

Definition: sectorial operator

- sector $\omega + \Sigma_{\frac{\pi}{2}+\alpha} \subset \rho(-A)$
- resolvent bound: $\|(\lambda + A)^{-1}\|_{op} \le \dfrac{M}{|\lambda - \omega|}$



Definition: sectorial operators have a natural semigroup

$$e^{-tA} = \begin{cases} \mathrm{Id}_X, & t = 0 \\ \dfrac{1}{2\pi i} \displaystyle\int_\gamma e^{t\lambda} (\lambda + A)^{-1} d\lambda, & t > 0 \end{cases}$$

Remark: $e^{-tA} \in \mathcal{L}(X)$ $\forall t \ge 0$ is a family of bounded linear operators

Properties $T(t) = e^{-tA}$

$$e^{-tA} = \begin{cases} \mathrm{Id}_X, & t=0 \\ \frac{1}{2\pi i} \int_\gamma e^{t\lambda} (\lambda + A)^{-1} d\lambda, & t>0 \end{cases}$$

1. $T(t)$ forms strongly continuous semigroup
   (a) $T(t+s) = T(t)T(s)$, $T(0) = \mathrm{Id}$
   (b) $\lim_{t \searrow 0} T(t)x = x \quad \forall x \in X$

2. $-A$ is generator of $T(t)$

$$\lim_{t \searrow 0} \frac{e^{-tA}x - x}{t} = -Ax \quad \forall x \in D(A)$$

3. smoothing property: observe: $(\lambda + A)^{-1}$ maps $X \to D(A)$ and $D(A^{k-1}) \to D(A^k)$

$$\Rightarrow \quad e^{-tA}x \in D(A^k) \quad \forall k \in \mathbb{N},\ t>0,\ x \in X$$

4. e^{-tA} solves Cauchy problem: $e^{-tA} \in C^\infty((0,\infty); \mathcal{L}(X))$

$$\frac{d^k}{dt^k} e^{-tA} = (-A)^k e^{-tA}, \quad t>0$$

5. e^{-tA} has analytic extension e^{-zA} to the sector $\Sigma_{\alpha-\varepsilon}$
   and $\lim_{z \to 0} T(z)x = x \quad \forall x \in X$ if $z \in \Sigma_\beta \quad \forall \beta < \alpha$.

Definition

- A is infinitesimal generator of analytic semigroup if
  - A satisfies 1. + 5. for some $a > 0$

$\mathcal{H}(X) = \bigcup_{a>0} \mathcal{H}_a(X)$ = set of generators of analytic semigroups

Theorem (Hille) TFAE

- $A \in \mathcal{H}(X)$
- A is sectorial
- $\{\mathrm{Re}\,\lambda \geq \omega\} \subset \rho(-A)$ and

$$\|(\lambda + A)^{-1}\|_{op} \leq \frac{M}{1+|\lambda|} \quad \forall \mathrm{Re}\,\lambda \geq \omega$$

Cauchy problem

$$\begin{cases} \frac{d}{dt} u(t) + Au(t) = 0 \\ u(0) = u_0 \end{cases}$$

has solution $u(t) = e^{-tA} u \in C([0,\infty); X) \cap C^\infty((0,\infty); D(A^k)) \quad \forall k$

if $u_0 \in D(A)$, then also $u \in C^1([0,\infty); X)$

Inhomogeneous Cauchy problem

$$\begin{cases} \frac{d}{dt} u(t) + A u(t) = f(t) \\ u(0) = u_0 \end{cases}$$

$\longrightarrow$ Variation-of-constants formula

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(s)\, ds$$

3. Thin-film equation as fixed-point problem

Recall that we had written

$$\partial_t h + \partial_x (h^n \partial_x^3 h) = 0$$

as the fixed-point problem

$$\begin{cases} \partial_t h + A[h] h = F(h) & \text{in } \Omega \\ B[h] = 0 & \text{on } \partial\Omega \\ h(0) = h_0 \end{cases}$$

where $A[g]h = g^n \partial_x^4 h$.

- Does $A[g]$ generate an analytic semigroup on $L^2(\Omega)$?

  If $g \in C([0,T] \times \Omega)$ with $g > 0$, then yes

  $\longrightarrow$ normally elliptic + Lopatinski-Shapiro condition

- Can we find a fixed point in $X(T) = \{ h \in C^1([0,T]; H^4(\Omega)) : h > 0 \}$?

  Take $u_0 \in H^4(\Omega) \hookrightarrow C^3(\bar{\Omega})$ with $u_0 > 0$.

  Then for $g \in X(T)$: $A[g] \in Lip([0,T]; \mathcal{H})$.

$\rightsquigarrow$ We may use variation-of-constants formula

+ Banach's fixed point theorem

$\longrightarrow$ obtain $T>0$ (even maximal)

$h \in C^1([0,T]; H^4(\Omega))$ with $h>0$ on $[0,T]\times\overline{\Omega}$

But: semigroup is infinitely smoothing

$\Rightarrow$ h is smooth on $(0,T)\times\overline{\Omega}$

(see more details e.g. in

Amann, Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems)

4. Outlook

- Strong solution concept fails, when solutions can become zero

  $\to$ weak solution theory

test $\partial_t h + \partial_x (h^n \partial_x^3 h) = 0$ with $\varphi \in C^\infty(\Omega)$ to obtain

$$\int_\Omega \partial_t h \, \varphi - \int_{\{h>0\}} h^n \partial_x^3 h \, \partial_x \varphi = 0$$

- Idea: construct weak solutions by regularisation

$$h^n \longrightarrow h^n + \varepsilon \qquad \text{"naive regularisation"}$$

$\hookrightarrow$ study limit points of h_ε via energy methods

Energy-dissipation mechanism

$$\frac{d}{dt} \int_\Omega |\partial_x h_\varepsilon(t)|^2 dx = - \int_\Omega (h^n + \varepsilon) \, |\partial_x^3 h_\varepsilon|^2 dx$$

is obtained by testing the equation with $\partial_x^2 h_\varepsilon$

Questions:

- non-negativity of solutions?
- uniqueness?
- what happens at points, where $h = 0$?

References

- for analytic semigroups

  A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems

- for quasilinear evolution equations, Lopatinski–Shapiro, ...

  H. Amann, Nonhomogeneous Linear and Quasilinear Elliptic and Parabolic Boundary Value Problems)