Naive global weak solutions to TFE

Based on: F. Bernis, A. Friedman (1990),
"Higher order nonlinear degenerate parabolic equations"

$$\frac{H}{L} \to 0$$

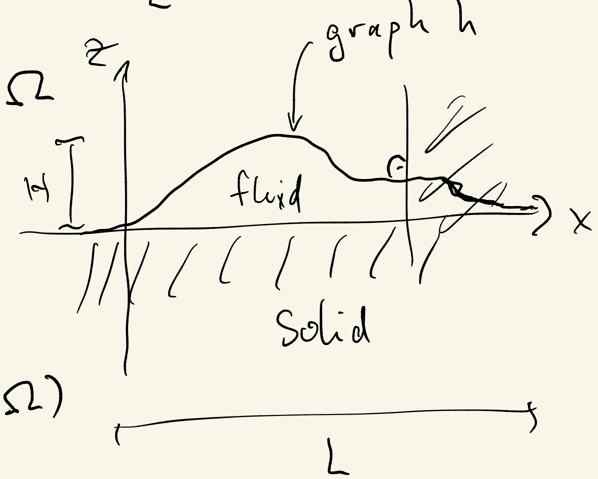
$$\partial_t h + \partial_x \left(|h|^n \partial_x^3 h \right) = 0 \quad \text{in } (0,T)\times\Omega$$

$$\Omega = (-a,a)$$

$$\partial_x h(t,\pm a) = 0$$

$$\partial_x^3 h(t,\pm a) = 0$$

$$h(0,x) = h_0(x) \quad \text{in } \Omega, \quad h_0 \in H^1(\Omega)$$



Remarks:

- We don't know anything about non-negativity
  $\rightarrow$ absolute value
- exponent n $(n \geq 1)$ ~ slip condition
  $n = 3$ ~ no-slip
- $\partial_x h(t,\pm a) = 0$ ~ zero contact angle
- $\partial_x^3 h(t,\pm a) = 0$ ~ mass conserv. / no-flux

Goal: Construct weak solutions to TFE

Main Problem: TFE is degenerate parabolic
as $h \downarrow 0$, parabolicity breaks down

Strategy: Step 1) Regularise the problem (and initial data) to obtain global classical solutions h_ε

$$\partial_t h_\varepsilon + \partial_x \Big(\underbrace{\left(|h_\varepsilon|^n + \varepsilon\right)}_{\geq \varepsilon > 0} \partial_x^3 h_\varepsilon \Big) = 0$$

Step 2) Find a limit $h_\varepsilon \to h$ as $\varepsilon \downarrow 0$ which is a weak solution.

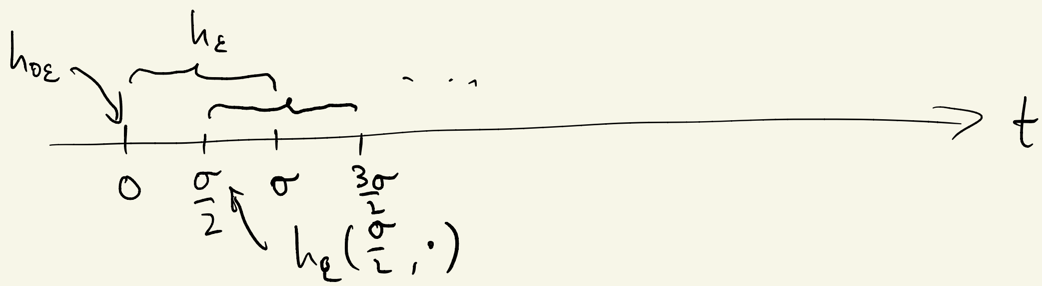
$$
(P_\varepsilon)\quad
\begin{cases}
\partial_t h_\varepsilon + \partial_x\big(\overbrace{(|h_\varepsilon|^n+\varepsilon)}^{\geq c_0>0\ \ \forall \varepsilon>0}\,\partial_x^3 h_\varepsilon\big) = 0 & \text{in } Q_T := (0,T)\times\Omega \\
\partial_x h_\varepsilon(t,\pm a) = \partial_x^3 h_\varepsilon(t,\pm a) = 0 & \forall t \\
h_\varepsilon(0,x) = h_{0\varepsilon}(x)
\end{cases}
$$

$\alpha \in (0,1)$

$$h_{0\varepsilon} \in C^{4+\alpha}(\Omega), \quad \partial_x h_{0\varepsilon}(\pm a) = \partial_x^3 h_{0\varepsilon}(\pm a) = 0$$

$$h_{0\varepsilon} \to h_0 \quad \text{as } \varepsilon \downarrow 0 \quad \text{in } H^1(\Omega)$$

$\to$ parabolic problem   analytic semigroup + fixed point



problem: time step may become small

A priori estimates: Test (P_ε) with $\partial_x^2 h_\varepsilon$

$$\int_\Omega \partial_t h_\varepsilon\, \partial_x^2 h_\varepsilon\, dx + \int_\Omega \partial_x^2 h_\varepsilon\, \partial_x\big((|h_\varepsilon|^n+\varepsilon)\,\partial_x^3 h_\varepsilon\big)\, dx = 0$$

$$\underbrace{\int_\Omega \partial_t \partial_x h_\varepsilon\, \partial_x h_\varepsilon\, dx}_{\frac{d}{dt}\underbrace{\frac{1}{2}\int_\Omega |\partial_x h_\varepsilon(t)|^2\, dx}_{=:E(t)}} + \int_\Omega (|h_\varepsilon|^n+\varepsilon)(\partial_x^3 h_\varepsilon)^2\, dx = 0$$

energy can only decrease

"energy dissipation equality"

$\Rightarrow \quad E(t) \le E(0) \qquad\qquad h_{0\varepsilon} \to h \quad \text{in} \quad H^1(\Omega)$

$$\|\partial_x h_\varepsilon(t)\|_{L^2(\Omega)} \le \|\partial_x h_{0\varepsilon}\|_{L^2(\Omega)} \longrightarrow \|\partial_x h_0\|_{L^2(\Omega)}$$

$$\|h_\varepsilon(t)\|_{H^1(\Omega)} \le C\,\|h_0\|_{H^1(\Omega)} \qquad \forall t, \varepsilon$$

$\Rightarrow$ global classical solutions to (P_ε) h_ε

Step 2) Find a limit $h_\varepsilon \to h$ as $\varepsilon \downarrow 0$, s.t. h is a weak solution to TFE

Sobolev embedding (1D) $\quad H^1(\Omega) \hookrightarrow C^{1/2}(\Omega)$

$$\|h_\varepsilon\|_{C^{1/8,1/2}_{t,x}(\overline{Q_T})} \le C\,\|h_0\|_{H^1(\Omega)}$$

(Arzela-Ascoli) $\Rightarrow$ $\quad \overline{\{h_\varepsilon : \varepsilon > 0\}} \subset\subset C(\overline{Q_T})$ compact

There is a subsequence h_ε and $h \in C^{1/8,1/2}_{t,x}(\overline{Q_T})$ s.t. $h_\varepsilon \to h$ uniformly in Q_T.

candidate for weak solution

Take test function φ with $\varphi = 0$ near $t=0$ and $t=T$

$$\iint_{Q_T} \varphi\, \partial_t h_\varepsilon \,dx\,dt + \iint_{Q_T} \varphi\, \partial_x\big((|h_\varepsilon| + \varepsilon)\, \underline{\partial_x^3 h_\varepsilon}\big)\,dx\,dt = 0$$

$$\underbrace{\iint_{Q_T} \partial_t \varphi \, h_\varepsilon \, dx\,dt}_{\to \iint_{Q_T} h \partial_t \varphi \, dx\,dt} + \iint_{Q_T} \partial_x \varphi \, (|h_\varepsilon|^n + \varepsilon) \, \partial_x^3 h_\varepsilon \, dx\,dt = 0$$

$$\varepsilon \iint \partial_x^3 h_\varepsilon \, \partial_x \varphi \, dx\,dt \to 0 \quad \text{as } \varepsilon \downarrow 0$$

$$\iint_{Q_T} |h_\varepsilon|^n \partial_x^3 h_\varepsilon \, \partial_x \varphi \, dx\,dt$$

$$= \iint_{\{|h_\varepsilon| \le \delta\}} \underbrace{|h_\varepsilon|^n}_{\le \delta^n} \partial_x^3 h_\varepsilon \, \partial_x \varphi \, dx\,dt$$

$$+ \iint_{\{|h_\varepsilon| > \delta\}} |h_\varepsilon|^n \partial_x^3 h_\varepsilon \, \partial_x \varphi \, dx\,dt$$

$$= O(\delta^{n/2}) + \iint_{\{|h_\varepsilon| > \delta\}} \underbrace{|h_\varepsilon|^n \partial_x^3 h_\varepsilon}_{=: F_\varepsilon} \underbrace{\partial_x \varphi}_{\in L^2} \, dx\,dt$$

$$\to \iint F \partial_x \varphi \, dx\,dt$$

$$\|F_\varepsilon\|^2_{L^2(Q_T)} = \iint_{Q_T} |h_\varepsilon|^{2n} (\partial_x^3 h_\varepsilon)^2 \, dx\,dt$$

$$\le \underbrace{\|h_\varepsilon\|^n_{L^\infty}}_{\le C} \underbrace{\iint_{Q_T} |h_\varepsilon|^n (\partial_x h_\varepsilon)^2 \, dx\,dt}_{\le C}$$

$$\Rightarrow \|F_\varepsilon\|_{L^2(Q_T)} \le C$$

$$\Rightarrow \exists \text{ subsequence } F_\varepsilon, \; \underline{\underline{\exists F}} \in L^2(Q_T) \text{ s.t.}$$

$F_\varepsilon \to F$ as $\varepsilon \downarrow 0$ in $L^2(Q_T)$

$$\iint_P F\, \partial_x \varphi \, dx\, dt$$

$$P = \overline{Q_T} \setminus (\{t=0\} \cup \{h=0\})$$

$$F|_P = |h|^n \partial_x^3 h$$

If $K \subset\subset P$ then

$\partial_x^k h_\varepsilon \to \partial_x^k h_\varepsilon$ uniformly in K

$\partial_t h_\varepsilon \to \partial_t h$ " "

$$\boxed{\iint_{Q_T} h\, \partial_t \varphi \, dx\, dt + \iint_P |h|^n \partial_x^3 h \, \partial_x \varphi \, dx\, dt = 0 \quad \forall \varphi \in Lip(\overline{Q_T}) \text{ with } \varphi = 0 \text{ near } t=0 \text{ and } t=T}$$

boundary cond. are satisfied in P

energy dissipation inequality for weak solution h

$$\frac{1}{2}\int_\Omega |\partial_x h(t)|^2 dx + \int_0^t \int_\Omega |h|^n (\partial_x^3 h)^2 \, dx\, ds \le \frac{1}{2}\int_\Omega |\partial_x h_0|^2 dx$$

$$\varepsilon \iint_{Q_T} \partial_x^3 h_\varepsilon \, \partial_x \varphi \, dx\,dt$$

$$= \varepsilon^{1/2} \iint_{Q_T} \underbrace{\varepsilon^{1/2} \partial_x^3 h_\varepsilon}_{} \, \underbrace{\partial_x \varphi}_{} \, dx\,dt$$

$$\leq \varepsilon^{1/2} \underbrace{\left(\iint_{Q_T} \varepsilon \left(\partial_x^3 h_\varepsilon\right)^2 dx\,dt \right)^{1/2}}_{} \; C$$