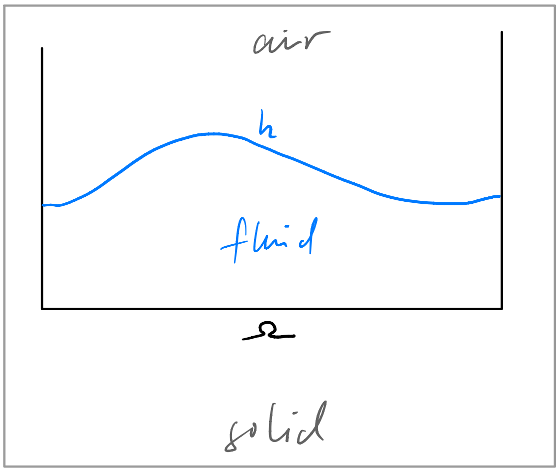
Entropy methods, non-negativity, and non-naive regularisation

(Bernis, Friedman 1990)

1. Recap & goals

thin-film equation



$$\begin{cases} h_t + (|h|^n h_{xxx})_x = 0 \quad \text{in } (0,T_0)\times \underset{(-a,a)}{\underset{\|}{\Omega}} \\ h_x(t, \pm a) = 0 \\ h_{xxx}(t, \pm a) = 0 \\ h(0,x) = h_0(x) \end{cases}$$

• Energy-dissipation inequality:

formally test with h_{xx}

$$\frac{d}{dt} \frac{1}{2}\int_\Omega h_x^2 \, dx \leq - \int_\Omega |h|^n h_{xxx}^2 \, dx \qquad (\leq 0)$$

$$\frac{d}{dt} \mathcal{E}[h](t) \leq - D[h](t)$$

OR

$$\frac{1}{2}\int_\Omega h_x(t)^2 \, dx + \iint_{\Omega_t} |h|^n h_{xxx}^2 \, dx\, ds \leq \frac{1}{2}\int_\Omega h_{0x}^2 \, dx$$

- Existence: via naive regularisation

solve the non-degenerate problem for $h = h_\varepsilon$

$$\begin{cases} h_t + \left((|h|^n + \varepsilon) h_{xxx} \right)_x = 0 \\ h(0,x) = h_{0\varepsilon}(x) \\ + \text{B.C.} \end{cases}$$

with $h_{0\varepsilon} \in C^{4+\alpha}(\Omega)$, $h_{0\varepsilon} \longrightarrow h_0$ in $H^1(\Omega)$

consider $\varepsilon \to 0$: $h_\varepsilon \longrightarrow h$ weak solution

$$\boxed{\begin{gathered} \iint_{Q_T} h \varphi_t \, dx dt + \iint_{P} |h|^n h_{xxx} \varphi_x \, dx dt = 0 \\ \forall \varphi \in Lip(\bar{Q}_{T_0}), \ \varphi = 0 \text{ near } t = 0 \text{ and } t = T_0 \end{gathered}}$$

Recall: $\| h_\varepsilon \|_{C^{1/8, 1/2}_{t,x}} \leq C$

- Goals:
  - non-negativity / positivity

$$h_0 \geq 0 \overset{?}{\Longrightarrow} h \geq 0$$

$$h_0 > 0 \overset{?}{\Longrightarrow} h > 0$$

  - introduce non-naive regularisation

2. Entropy methods and non-negativity

Assume $h_0 \in H^1(\Omega)$, $\underline{\underline{h_0 \geq 0}}$ throughout.

. Formal derivation of entropy estimate

$\longrightarrow$ "test cleverly"

$$\boxed{\begin{aligned} &\text{Define} \quad g_\varepsilon(s) = -\int_s^A \frac{1}{|r|^n+\varepsilon}\,dr, \quad A > \max h_\varepsilon \\ &\qquad\qquad G_\varepsilon(s) = -\int_s^A g_\varepsilon(r)\,dr \end{aligned}}$$

$$\Rightarrow \quad G_\varepsilon'(s) = g_\varepsilon(s), \quad g_\varepsilon'(s) = \frac{1}{|s|^n+\varepsilon}$$

$$g_\varepsilon(s) \leq 0, \quad G_\varepsilon(s) \geq 0 \qquad s \leq A$$

Let $G_0(s) := \lim_{\varepsilon \to 0} G_\varepsilon(s)$, $T \in (0, T_0)$

Test with $G_0'(h)$: Formal $\qquad G_0''(s) = \frac{1}{|s|^n}$

$$\iint_{Q_T} h_t\, G_0'(h) + (|h|^n h_{xxx})_x\, G_0'(h)\, dx\,dt = 0$$

$$\iint_{Q} \frac{d}{dt} G_0(h) - \cancel{|h|^n} h_{xxx} \frac{h_x}{\cancel{|h|^n}}\, dx\,dt = 0$$

$$\rightsquigarrow \int_\Omega G_0(h(T))\,dx + \iint_{Q_T} h_{xx}^2\, dx\,dt = \int_\Omega G_0(h_0)\,dx$$

What if $h = 0$? $\longrightarrow$ Work with $\frac{1}{|h|^n+\varepsilon}$

Note that

$$G_0(s) = \begin{cases} C_0 + C_1 s - C_2 s^{2-n} & \text{if } 1 < n < 2 \\ \log \frac{C_3}{s} + \frac{s}{C_3} - 1 & \text{if } n = 2 \\ C_4 s^{2-n} - C_5 + s C_6 & \text{if } n > 2 \end{cases}$$

· extend assumptions on h_0: $h_0 \in H^1(\Omega)$, $h_0 \geq 0$,

$$(A) \begin{cases} \int_\Omega |\log h_0| \, dx < \infty & \text{if } n = 2 \\ \int_\Omega h_0^{2-n} \, dx < \infty & \text{if } 2 < n < 4 \\ h_0 > 0 & \text{if } n \geq 4 \end{cases}$$

$\leftarrow$ $\{h_0 = 0\}$ is zero set (for $n = 2$ and $2 < n < 4$)

$\leftarrow$ not stronger (for $n \geq 4$): $h_0^{2-n} \in L^1$, $\frac{1}{2}$ Hölder-cont

<u>Theorem 1</u> Assuming (A)

(i) if $n > 1$ then $h \geq 0$ in Q_{T_0}

(ii) if $n \geq 2$ then $\operatorname{meas}\{h = 0\} = 0$

if $n = 2$ $\quad \int_\Omega |\log(h(t))| \, dx \leq C \quad \forall t$

if $n > 2$ $\quad \int_\Omega h^{2-n}(t) \, dx \leq C \quad \forall t$

(iii) if $n \geq 4$, then $h > 0$ in $\overline{Q_{T_0}}$

Proof (i) and (iii)

Step 1 bound initial entropy of h_ε

Choose $h_{0\varepsilon} \searrow h_0$ in $H^1(\Omega)$ $\Rightarrow$ $h_{0\varepsilon} > 0$

$G_\varepsilon \le G_0$, (A) $\Rightarrow$

$$\int_\Omega G_\varepsilon(h_{0\varepsilon})\,dx \le C \quad \text{unif} \qquad (B_0)$$

Step 2 derive entropy equality for h_ε

Test $\quad h_{\varepsilon t} + \left((|h_\varepsilon|^n + \varepsilon)\, h_{\varepsilon xxx}\right)_x = 0$

with $\quad g_\varepsilon(h_\varepsilon)$

$$0 = \iint_{Q_T} h_{\varepsilon t}\, g_\varepsilon(h_\varepsilon) + \left((|h_\varepsilon|^n + \varepsilon)\, h_{\varepsilon xxx}\right)_x g_\varepsilon(h_\varepsilon)\,dx\,dt$$

$G_\varepsilon' = g_\varepsilon$

$$= \iint_{Q_T} \frac{d}{dt} G_\varepsilon(h_\varepsilon)\,dx - \cancel{(|h_\varepsilon|^n + \varepsilon)}\, h_{\varepsilon xxx} \frac{h_{\varepsilon x}}{\cancel{|h_\varepsilon|^n + \varepsilon}}\,dx\,dt$$

$\Rightarrow$ entropy equality

$$\int_\Omega G_\varepsilon(h_\varepsilon(T))\,dx + \iint_{Q_T} h_{\varepsilon xx}^2\,dx\,dt = \int_\Omega G_\varepsilon(h_{0\varepsilon})\,dx$$

use (B_0) to obtain

$$\int_\Omega G_\varepsilon(h(T))\,dx \le C \qquad (B_T)$$

"the entropy remains bounded"

Step 3 prove $h \geq 0$

Assume by contradiction

$$\exists\ (x_0, t_0) \in Q_{T_0} \quad \text{s.t.} \quad h(x_0, t_0) < 0$$

As $h_\varepsilon \to h$ unif, h is cont $\Rightarrow$

$\exists\ \delta > 0,\ \varepsilon_0 > 0$ s.t.

$$h_\varepsilon(x, t_0) < -\delta \quad \text{for} \quad x \in B_\delta(x_0),\ \varepsilon < \varepsilon_0$$

For $x \in B_\delta(x_0)$

$$G_\varepsilon(h_\varepsilon(x, t_0)) = -\int_{h_\varepsilon(x,t_0)}^{A} g_\varepsilon(s)\, ds$$

$$\overset{g_\varepsilon(s) \leq 0}{\geq} -\int_{-\delta}^{0} g_\varepsilon(s)\, ds$$

$$\xrightarrow{\text{mct}} -\int_{-\delta}^{0} g_0(s)\, ds$$

where $g_0(s) = \lim_{\varepsilon \to 0} g_\varepsilon(s) = \lim_{\varepsilon \to 0} -\int_s^A \frac{1}{|r|^n + \varepsilon}\, dr$

But $g_0(s) = -\infty$ if $s \leq 0$

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{B_\delta(x_0)} G_\varepsilon(h_\varepsilon(x, t_0))\, dx = \infty \quad \lightning \ (B_T)$$

Step 4 prove $n \geq 4$, $h_0 > 0 \Rightarrow h > 0$

(B_T) + Fatou : $\int_\Omega G_0(h(T))\, dx \leq C$, $(\tilde{B}_T)$

$\sim h^{2-n}$

Assume by contradiction $\exists (x_0, t_0) \in \overline{Q}_{T_0}$: $h(x_0,t_0) = 0$

By $\frac{1}{2}$ - Hölder in space

$$h(x, t_0) < K\, |x - x_0|^{\frac{1}{2}}$$

$\Rightarrow$

$$\int_\Omega h^{2-n}(x, t_0)\, dx > K \int_\Omega |x - x_0|^{\frac{2-n}{2}}\, dx = \infty \quad \text{if } n \geq 4$$

$\lightning$ to $(\tilde{B}_T)$

$\square$

Remark if $n \geq 4$ $h > 0$ is unique pos. sol.

Theorem 2

$\forall\, h_0 \geq 0 \;\exists$ w.s. $h \geq 0$

(We do not need (A), we add a step of approx)

Proof sketch

- let $\tilde{h}_{0\delta} = h_0 + \delta$

$\Longrightarrow$ $\tilde{h}_{0\delta}$ satisfies (A) $\forall\, \delta > 0$ $\star$

- solve TFE for $\tilde{h}_{0\delta}$

$$\begin{cases} (\tilde{h}_\delta)_t + \left((|\tilde{h}_\delta|^n + \varepsilon)\, \tilde{h}_{\delta xxx}\right)_x = 0 \\ \tilde{h}_\delta(0) = \tilde{h}_{0\delta} \end{cases}$$

+ BC

$\longrightarrow$ using unif bounds

$$\tilde{h}_{\delta\varepsilon} \xrightarrow{\varepsilon \to 0} \tilde{h}_\delta \geq 0 \quad \text{bc of } \star$$

$$\tilde{h}_\delta \xrightarrow{\delta \to 0} h \geq 0 \quad \text{w.s. TFE}$$

$\square$

3. Non-naive regularisation

$$\begin{cases} h_t + (|h|^n h_{xxx})_x = 0 \\ h(0) = h_0 \\ \text{B.C.} \end{cases}$$

previously : h_ε could be negative

now : h_ε is positive

Strategy : choose $h_{0\varepsilon} = h_0 + \varepsilon > 0$

and replace $|h|^n$ by $m_\varepsilon(h_\varepsilon)$ with

$$m_\varepsilon(s) = \frac{s^4 |s|^n}{\varepsilon |s|^n + s^4}$$

consider
$$\begin{cases} h_t + (m_\varepsilon(h) h_{xxx})_x = 0 \\ h(0) = h_{0\varepsilon} \\ + \text{B.C.} \end{cases}$$

$m_\varepsilon(s) \sim |s|^n$ if $n \geq 4$

$$m_\varepsilon(s) \sim \begin{cases} |s|^n & \text{if } 0 < \zeta_\varepsilon \leq s \\ \frac{s^4}{\varepsilon} & \text{if } 0 \leq s < \zeta_\varepsilon \end{cases}$$

h_ε admits entropy bounds using

$$g_\varepsilon(s) = -\int_s^A \frac{1}{m_\varepsilon(r)}\,dr, \qquad G_\varepsilon(s) = -\int_s^A g_\varepsilon(r)\,dr$$

By a generalization of Theo 1 (iii)

$h_\varepsilon > 0$ (unique pos sol, smooth)

Take $\varepsilon \to 0$: $h_\varepsilon \to h$ ws TFE

$\longrightarrow$

- prop of support
- positivity
- regularity