TRAVELING WAVES FOR THIN-FILM EQUATIONS

REPETITION

The thin-film equation is given by

$$\boxed{h_t + (h^n h_{xxx})_x = 0}$$

So far we have studied (on bounded domain Ω with $h_x = h_{xxx} = 0$ on $\partial\Omega$)

....
- Mathematical modeling (lubrication approximation)
- Local well-posedness (semigroup theory)
- Existence of non-negative global weak solutions (energy methods)

TRAVELING WAVES

A traveling wave for the thin film equation is a solution

$$h(t,x) = H(x-ct)$$

where $c \in \mathbb{R}$ is the wave speed

- $c > 0$: right-going
- $c < 0$: left-going
- $c = 0$: steady state

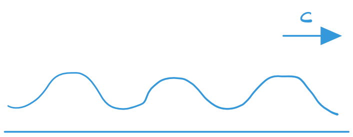
The function H solves the ODE

$$-cH' + (H^n H''')' = 0$$

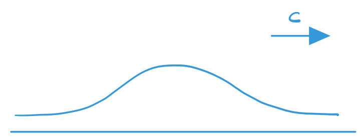
PRELIMINARY THOUGHTS ON TRAVELING WAVES

What kind of traveling waves do we expect / are interesting in the context of thin-film equations (parabolic!)?

Speaking about traveling waves one often thinks about ...



periodic



solitary ($H \to 0$ as $|x| \to \infty$)

↯ IMPOSSIBLE FOR THE THIN FILM EQUATION DUE TO DISSIPATION

Recall the energy inequality (testing the thin-film eq. with h_{xx})

$$\boxed{\frac{d}{dt}\frac{1}{2}\int (h_x)^2\,dx = -\int h^n\,(h_{xxx})^2\,dx}$$

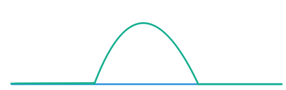
Assume that $h(t,x) = H(x-ct) \geq 0$ is a periodic ($\int_{\mathbb{T}}$) or solitary ($\int_{\mathbb{R}}$) solution

$\Rightarrow \qquad 0 = -\int H^n\,(H''')^2\,dx$

$\Rightarrow \qquad$ EITHER $H \equiv 0$ OR $H''' \equiv 0$ ($\overset{\text{eq.}}{\Rightarrow}$ $c = 0$)

If $H''' = 0$

periodic



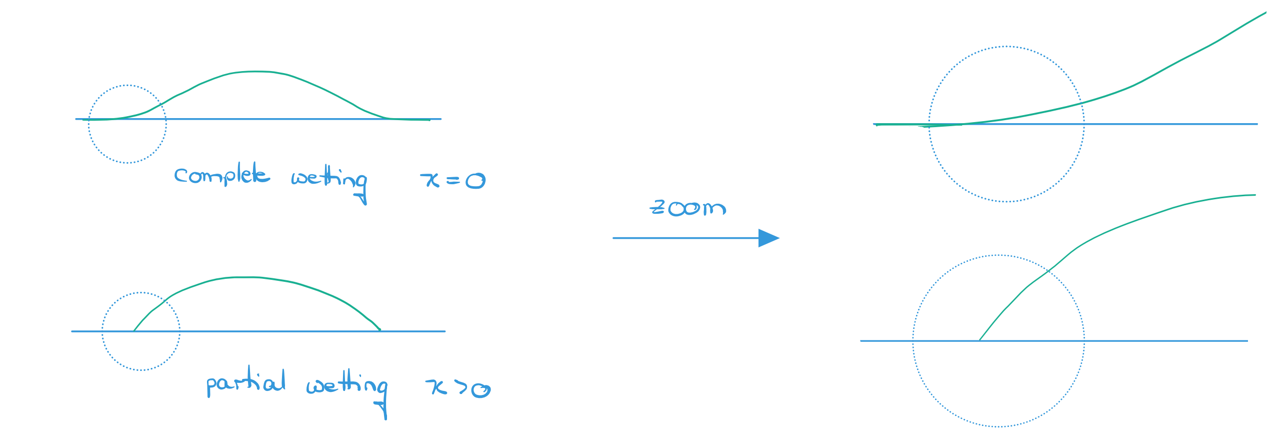
solitary

$\boxed{c = 0}$

steady states

CONCLUSION: There exist no nontrivial periodic or solitary traveling wave solutions with $c \neq 0$ for the thin-film equation.

MOTIVATION (what are we looking for?)



Chiricotto & Giacomelli (2011)

> In the case of a spreading droplet, the local behavior near the contact line is that of an advancing traveling wave, whose profile is determined by "matching" it to the bulk region. This procedure has been followed in the past by many authors [15–20] in order to obtain qualitative information on the macroscopic dynamics. In all of these papers, the matching condition selects the solution to (7) which displays the "linear" behavior at infinity.

BEHAVIOR CLOSE TO CONTACT POINT: we look for traveling wave solutions so that

$$h(t,x) = H(x-ct)$$

$$\begin{cases} c\,H = H^n H''' & \text{on } (0,\infty) \\ H(0) = 0 \quad H'(0) = \varkappa \geq 0 \quad \lim_{x\to\infty} H''(x) = 0 \end{cases}$$

- $c > 0$: receding
- $\boxed{c < 0 \ : \ \text{exceeding}}$

THEOREM (Existence of exceeding traveling waves)

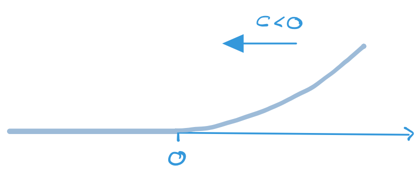
Let $n \in (1,3)$, $c < 0$ and $\kappa \geq 0$. Then there exists a unique, global traveling-wave solution $h(t,x) = H(x-ct)$ of the thin-film eq. where

$$H \in C^1([0,\infty),[0,\infty)) \cap C^3(\{H>0\})$$

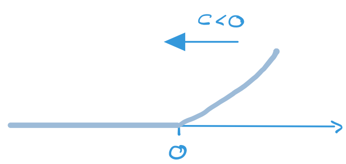
satisfies $(*)$ and has subquadratic growth $\left(\lim_{x\to\infty} H''(x) = 0\right)$

CHIRICOTTO, GIACOMELLI (2011)

PROOF



complete wetting $\kappa = 0$



partial wetting $\kappa > 0$

$$\begin{cases} H''' = c\, H^{1-n} \\ H(0) = 0 \qquad H'(0) = \kappa \qquad \lim_{x\to\infty} H''(x) = 0 \end{cases}$$

Step 1 : Approximative system

For any $\varepsilon > 0$ consider

$$(P_\varepsilon) \quad \begin{cases} H_\varepsilon''' = c\, H_\varepsilon^{1-n} \quad \text{in } (0, \tfrac{1}{\varepsilon}) \\ H_\varepsilon(0) = \varepsilon \qquad H_\varepsilon'(0) = \kappa \qquad H_\varepsilon''(\tfrac{1}{\varepsilon}) = 0 \end{cases}$$

Linear problem

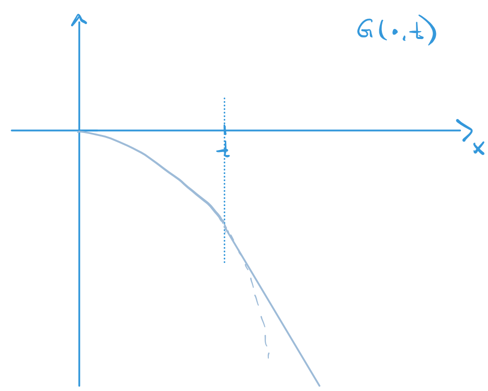
$$(PL_\varepsilon) \quad \begin{cases} H_\varepsilon''' = f & \text{in } (0, \frac{1}{\varepsilon}) \\ H_\varepsilon(0) = \varepsilon & H_\varepsilon'(0) = \kappa \qquad H_\varepsilon''(\frac{1}{\varepsilon}) = 0 \end{cases}$$

Then for any $f \in C([0,\infty))$ the function

$$H_\varepsilon(x) = \varepsilon + \kappa x + \int_0^{1/\varepsilon} G(x,t) f(t)\, dt$$

is a solution of (PL_ε), where G is the Green's function

$$G(x,t) = \begin{cases} -\frac{x^2}{2} & \text{if } x \le t \\ \frac{t^2}{2} - xt & \text{if } x > t \end{cases}$$



- $G(0,t) = 0$
- $G_x(0,t) = 0$
- $G_{xx}(x,t) = \begin{cases} -1 & \text{if } x \le t \\ 0 & \text{if } x > t \end{cases}$
- $G_{xxx}(x,t) = \delta(x-t)$

Nonlinear problem: Apply Schauder's fixed point argument

Let $S := \{ g \in C([0,\frac{1}{\varepsilon}]) \mid \varepsilon \le g \le \Pi_\varepsilon \}$ $\qquad \Pi_\varepsilon = \varepsilon + \frac{\kappa}{\varepsilon} - c\,\varepsilon^{-n-2}$

$\Rightarrow$ $S \subset C([0,\frac{1}{\varepsilon}])$ closed, bounded, convex subset

Define the fixed point problem

$$F : S \to C([0,\tfrac{1}{\varepsilon}])$$

$$F(g)(x) := \varepsilon + \kappa x + c \int_0^{\frac{1}{\varepsilon}} G(x,t)\, g^{1-n}(t)\, dt$$

If F has a fixed point H, then

$$H(x) = \varepsilon + \kappa x + c \int_0^{\frac{1}{\varepsilon}} G(x,t)\, H^{1-n}(t)\, dt$$

is a solution of (PL_ε).

To show:

a/ $\mathcal{F}(q) \in \mathcal{S}$ for any $q \in \mathcal{S}$

b/ $\mathcal{F}(\mathcal{S})$ is relatively compact in $\mathcal{S}$

$\Big\}\overset{\text{Schauder}}{\Longrightarrow}$ $\exists$ fixed point

(uniqueness a priori not known)

a/ Let $q \in \mathcal{S}$.

$$\mathcal{F}(q)(x) = \varepsilon + \varkappa x + c\int_0^{1/\varepsilon} G(x,t)\, q^{1-n}(t)\, dt \qquad \overset{q \geq \varepsilon}{\Rightarrow} \mathcal{F}(q) \in C([0,\tfrac{1}{\varepsilon}])$$

- $\mathcal{F}(q)(x) \geq \varepsilon + \varkappa x \qquad (c\, G(r,t) \geq 0,\ q \geq \varepsilon)$

$\to$ For $c > 0$ this could not be guaranteed

- $$(\mathcal{F}(q))'(x) = \varkappa + c\int_0^{1/\varepsilon} \underbrace{G_x(x,t)}_{\leq 0}\, q^{1-n}(t)\, dt \geq \varkappa$$
- $$(\mathcal{F}(q))''(x) = c\int_0^{1/\varepsilon} \underbrace{G_{xx}(x,t)}_{\leq 0}\, q^{1-n}(t)\, dt \geq 0$$

In addition

$$(\mathcal{F}(q))''(x) = \int_0^{1/\varepsilon} \underbrace{c\, G_{xx}(x,t)}_{\in c[-1,0] \geq 0} \underbrace{q^{1-n}(t)}_{\overset{q\geq\varepsilon}{\underset{n\geq 1}{\leq}} \varepsilon^{1-n}} dt \leq -\int_0^{1/\varepsilon} c\,\varepsilon^{1-n} dt = -c\,\varepsilon^{-n}$$

$$\Rightarrow (\mathcal{F}(q))'(x) = \underbrace{(\mathcal{F}(q))'(0)}_{=\varkappa} + \int_0^x (\mathcal{F}(q))''(y)\, dy$$

$$\leq \varkappa - c\,x\,\varepsilon^{-n} \overset{x \leq \frac{1}{\varepsilon}}{\leq} \varkappa - c\,\varepsilon^{-n-1}$$

$$\Rightarrow (\mathcal{F}(q))(x) = \underbrace{(\mathcal{F}(q))(0)}_{=\varepsilon} + \int_0^x (\mathcal{F}(q))'(y)\, dy$$

$$\leq \varepsilon + \int_0^x \varkappa - c\,\varepsilon^{-n-1}\, dy$$

$$\overset{x \leq \frac{1}{\varepsilon}}{\leq} \varepsilon + \frac{\varkappa}{\varepsilon} - c\,\varepsilon^{-n-2} = \Pi_\varepsilon$$

$\Rightarrow$ $\boxed{\mathcal{F}(q) \in \mathcal{S}}$

b/ $\mathcal{F}(S)$ bounded subset of $C^1([0,\frac{1}{\varepsilon}])$

$\overset{\text{Arzela-}}{\underset{\text{Ascoli}}{\Longrightarrow}}$ $\mathcal{F}(S)$ rel. compact in $C([0,\frac{1}{\varepsilon}])$

$\overset{\text{Schauder}}{\Longrightarrow}$ There exists $H \in S$ such that

$$\mathcal{F}(H) = H = \varepsilon + \kappa x + c\int_0^{1/\varepsilon} G(x,t) H^{1-n}(t)\, dt$$

solves (P_ε)

Step 2 : Passing to the limit $\varepsilon \to 0$

$\longrightarrow$ uniform bounds on H_ε (fails for $n \geq 3$)

Step 3 : Uniqueness

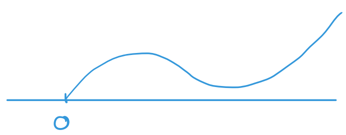
REMARK

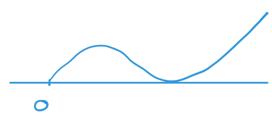
For $n=1$, there exist a global traveling-wave solution for any $c>0$. There exist no non-negative global traveling-wave solution for $c<0$.

If $n=1$, then $H''' = c H^{1-n} = c$

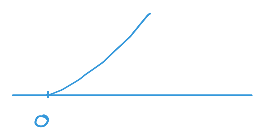
$$\Rightarrow \quad H(x) = \frac{c}{6}x^3 + bx^2 + \varkappa x \qquad (H(0)=0,\ H'(0)=\varkappa \geq 0)$$

- For $c>0$:

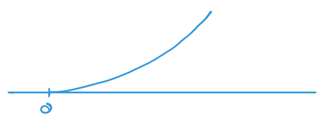




depending on b



$\varkappa > 0$



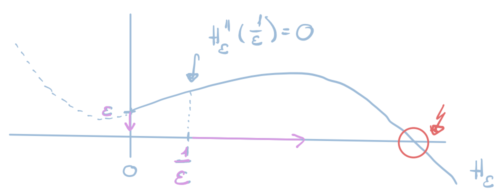
$\varkappa = 0$

- For $c<0$: $\lim_{x\to\infty} H(x) = -\infty$ ⚡

Recall that the approximate problem

$$\begin{cases} H_\varepsilon''' = c H_\varepsilon^{1-n} & \text{in } (0,\frac{1}{\varepsilon}) \\ H_\varepsilon(0) = \varepsilon, \quad H_\varepsilon'(0) = \varkappa, \quad H_\varepsilon''(\frac{1}{\varepsilon}) = 0 \end{cases}$$

has a unique solution for ALL $n \geq 1$



$\boxed{n=1}$