08 FEB 2023

SEMINAR "ASYMPTOTIC MODELS IN FLUID DYNAMICS"

SUMMARY OF THE FIRST SEASON

FURTHER PROPERTIES OF SOLUTIONS

OUTLOOK

Note that the below mentioned results are not rigorous / complete but rather meant to give an idea.
See the papers for rigorous results

I. THE STORY SO FAR

1. MODELLING:

NAVIER-STOKES $\xrightarrow{\varepsilon = \frac{H}{L} \to 0}$ THIN-FILM EQUATION

$$\boxed{\begin{aligned} \rho(\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u}) &= \operatorname{div} \sigma(\vec{u}, p) \\ \operatorname{div} \vec{u} &= 0 \\ &+ \text{BC} \\ h_t + u h_x = v, &\quad z = h(t,x) \end{aligned}}$$

$$\boxed{\begin{aligned} -p_x + (\mu u_z)_z &= 0 \\ p_z &= 0 \\ &+ \text{BC} \\ h_t + \left(\int_0^h u(t,x,z)\,dz \right)_x &= 0 \end{aligned}}$$

2. SHORT-TIME EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS

$$\text{PDE} \xrightarrow[\quad]{\mathcal{A}(v)\in\mathcal{L}(H^4_B;L_2) \qquad \mathcal{A}(v)h = v^n\partial_x^4 h} \text{ABSTRACT CAUCHY PROBLEM}$$

$$\boxed{\begin{array}{ll} h_t + (h^n h_{xxx})_x = 0, & t>0,\ x\in\Omega \\ h_x = h_{xxx} = 0, & t>0,\ x\in\partial\Omega \\ h(0,x) = h_0(x) > 0, & x\in\Omega \end{array}}$$

$$\boxed{\begin{array}{ll} h' + \mathcal{A}(h)h = F(h), & t>0 \\ h(0) = h_0 > 0 & \end{array}}$$

"PARABOLIC THEORY"

$$\boxed{\begin{array}{l} \exists\, T>0 \text{ and unique sol.} \\ h\in C([0,T];L_2(\Omega))\cap C^1((0,T);L_2(\Omega)) \end{array}}$$

3. EXISTENCE OF GLOBAL NON-NEGATIVE WEAK SOLUTIONS

PDE $\longrightarrow$ Regularised PDE

$$\boxed{\begin{array}{ll} h_t + (h^n h_{xxx})_x = 0, & t>0, x\in\Omega \\ h_x = h_{xxx} = 0, & t>0, x\in\partial\Omega \\ h(0,x) = h_0(x) > 0, & x\in\Omega \end{array}}$$

$$\boxed{\begin{array}{l} h^\varepsilon_t + \left((h^\varepsilon+\varepsilon)^n h^\varepsilon_{xxx}\right)_x = 0 \\ h^\varepsilon_x = h^\varepsilon_{xxx} = 0 \\ h^\varepsilon(0,\cdot) = h_{0\varepsilon} \longrightarrow h_0 \text{ in } H^1(\Omega) \end{array}}$$

Energy dissipation

$$\frac{1}{2}\int_\Omega |h^\varepsilon_x|^2 + \int_0^t\int_\Omega (h^\varepsilon+\varepsilon)^n |h^\varepsilon_{xxx}|^2 \leq \frac{1}{2}\int_\Omega |h_{0\varepsilon x}|^2$$

unif. estimates

$\forall$ suitable test functions φ:

$$\int_0^T\int_\Omega h\varphi_t + \iint_{\{h>0\}} h^n h_{xxx}\varphi_x = 0$$

4. ENTROPY, NON-NEGATIVITY AND NON-NAIVE REGULARISATION

test with g_ε, $\varepsilon \downarrow 0$:

$$\int_\Omega G_0(h(T)) + \iint h_{xx}^2 = \int G_0(h_0)$$

ENTROPY $\longrightarrow$ THEOREM [BF90]

ENTROPY

$$g_\varepsilon(s) = -\int_s^A \frac{1}{|r|^n+\varepsilon}\,dr \leq 0$$

$$G_\varepsilon(s) = -\int_s^A g_\varepsilon(r)\,dr \geq 0$$

THEOREM [BF90]

Assume $h_0 \in H^1(\Omega)$, $h_0 \geq 0$.

$n > 1$, $\int_\Omega |\log h_0| < \infty \implies h \geq 0$

$n \geq 2$, $\int_\Omega h_0^{2-n} < \infty \implies |\{h = 0\}| = 0$

$n \geq 4$, $h_0 > 0 \implies h > 0$

5. TRAVELLING-WAVE SOLUTIONS:

$$\boxed{\begin{aligned} &h(t,x) = H(x-ct) \geq 0 \\ &-cH' + (H^n H''')' = 0 \end{aligned}}$$

ODE (TW ansatz)

$$(*)\quad \boxed{\begin{aligned} &cH = H^n H''' \\ &H(0) = 0, \quad H'(0) = \delta \geq 0, \quad \lim_{x\to\infty} H''(x) = 0 \end{aligned}}$$

$$\xrightarrow{\text{Schauder for } (P_\varepsilon)}$$

THEOREM [CG 2011]

Let $n \in (1,3)$, $c < 0$, $\delta \geq 0$. Then: $\exists!$ sol.

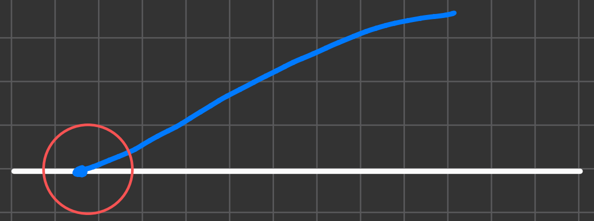
$$H \in C^1(\mathbb{R}_+; \mathbb{R}_+) \cap C^3(\{H > 0\})$$

to $(*)$.

II. FURTHER PROPERTIES OF SOLUTIONS

1. UNIQUENESS OF WEAK SOLUTIONS (only $n \geq 4$, $E_{rel} := \frac{1}{2}\int_\Omega |u_x(t) - v_x(t)|^2$)
2. NO-SLIP PARADOX
3. PROPAGATION OF SUPPORT
4. FINITE SPEED OF PROPAGATION
5. WAITING-TIME PHENOMENA
6. LONG-TIME BEHAVIOUR : $h(t,x) \longrightarrow \frac{1}{|\Omega|}\int_\Omega h_0 \, dx$ unif.
7. NON-NEWTONIAN FLUIDS

2. NO-SLIP PARADOX [Dussan-Davis 74], [Huh-Scriven 71]



NO-SLIP CONDITION

$$u = 0, \qquad z = 0$$

$\longrightarrow$ NO-SLIP PARADOX:

Heuristically:

- Consider $\displaystyle\iint_{\{h=0\}} h^n h_{xxx}^2$
- Assume $h \sim x^{3/n}$

Then:

$$h^n h_{xxx}^2 \sim x^3 x^{2(3/n-3)} = x^{3+\frac{6}{n}-6} = x^{\frac{6}{n}-3}, \qquad \frac{6}{n}-3 > -1 \iff n < 3$$

POSSIBLE REMEDIES

- Navier-slip condition:

$$u = \lambda^{3-n} h^{n-2} u_z, \qquad z = 0$$

$$\longrightarrow \quad h_t + \left(\left[\lambda^{3-n} h^n + h^3\right] h_{xxx}\right)_x = 0$$

- shear-thinning rheology
- equation with additional potential

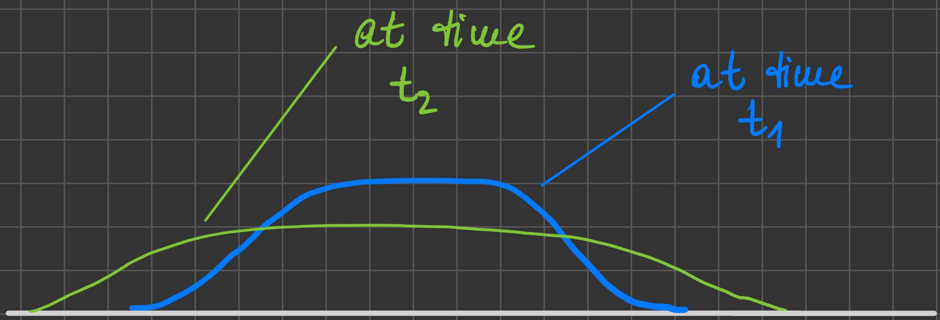
3. PROPAGATION OF SUPPORT

THEOREM [BF 90]

Let $n \geq 4$, $u \geq 0$ non-neg. weak sol.

Then $\forall\ 0 \leq t_1 \leq t_2 < T$:

$$\mathrm{supp}\,(u(t_1,\cdot)) \subseteq \mathrm{supp}\,(u(t_2,\cdot))$$



4. FINITE SPEED OF PROPAGATION

- rules out instantaneous complete wetting

THEOREM [Bernis 96 a, b]

Let $n \in (0,3)$ and h a weak solution. If

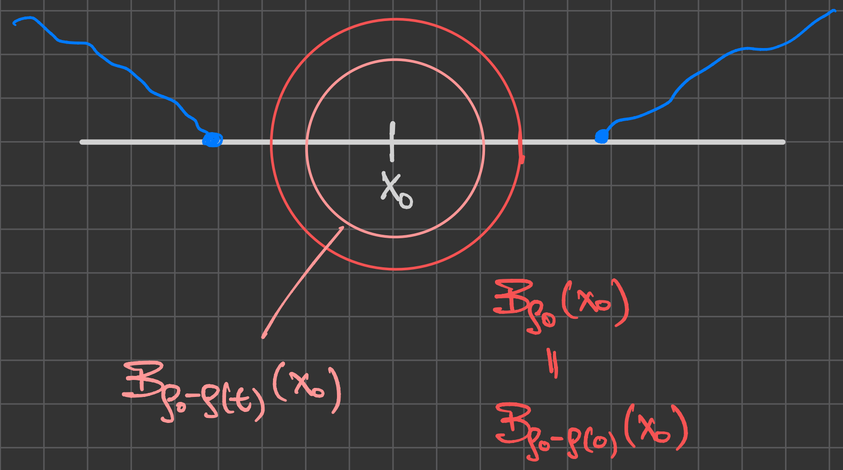
$$h(t_0,\cdot)=0 \quad \text{in } B_{\rho_0}(x_0)\subset\Omega,$$

then h has FSP:

$\exists\, t^* \in (0,\infty)$, $\exists$ nondecreasing function $\rho \in C([0,t^*);\mathbb{R}_{\geq 0})$ with $\rho(0)=0$ s.t.

$$h(t,\cdot)=0 \quad \text{in } B_{\rho_0-\rho(t)}(x_0)$$

for all $t \in (t_0, t_0+t^*)$.



5. WAITING-TIME PHENOMENON

- Speed of propagation can become slower / zero / negative for a certain time $\longrightarrow$ waiting time phenomenon
- Roughly: growth condition on initial value near contact point
  $\Longrightarrow$ occurence of WTP

THEOREM [Bernis 96 ; Dal Passo, Giacomelli, Grün 01]

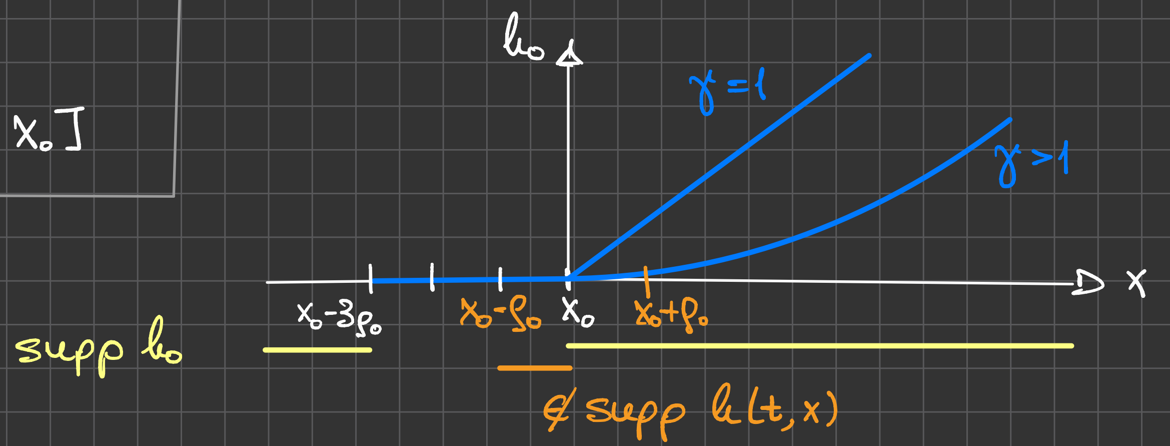
Let $n \in (0,3)$, $\gamma \geq \frac{4}{n}$ and h entropy solution with h_0 s.t.

$$h_0 = 0, \quad x \in [x_0 - 3\rho_0, x_0], \qquad \text{and} \qquad h_0 \sim |x - x_0|^\gamma \text{ at } x_0$$

Then $\exists\, t^* > 0$:

$$h(t,\cdot) = 0 \quad \text{in} \quad [x_0 - \rho_0, x_0]$$

for all $t \in (0, t^*)$.



6. NON-NEWTONIAN FLUIDS

NEWTONIAN FLUIDS

- $\mu \equiv \mu_0$ const.
- $\sigma(\epsilon) = -pI + 2\mu_0 \epsilon$
- Cons. of momentum:

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \mu \Delta \vec{u}$$

- Thin-film equ.

$$h_t + (h^n h_{xxx})_x = 0$$

NON-NEWTONIAN FLUIDS

- $\mu = \mu(|\epsilon|)$ shear-dep.
- $\sigma(\epsilon) = -pI + 2\mu(|\epsilon|)\,\epsilon$
- Cons. of momentum

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \operatorname{div}(2\mu(|\epsilon|)\epsilon)$$

- Thin-film equations:

POWER LAW: $\mu(|\epsilon|) = \mu_0 |\epsilon|^{\frac{1}{\alpha}-1}, \; \alpha > 0$

$$h_t + \left(h^{\alpha+2} |h_{xxx}|^{\alpha-1} h_{xxx}\right)_x = 0$$

ELLIS LAW: $\dfrac{1}{\mu(|\epsilon|)} = \dfrac{1}{\mu_0}\left(1 + \left|\dfrac{\mu(|\epsilon|)\epsilon}{\tau_{1/2}}\right|^{\alpha-1}\right), \; \alpha > 1$

$$h_t + \left(h^3 h_{xxx} + h^{\alpha+2} |h_{xxx}|^{\alpha-1} h_{xxx}\right)_x = 0$$