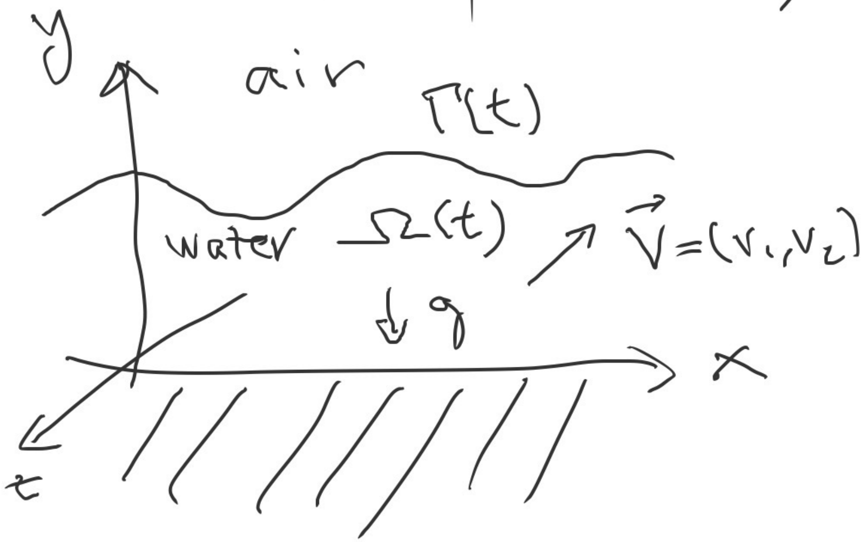
The water wave problem

(and its many shapes)



- Incompressible
  const. density
- Inviscid

Incompressible Euler

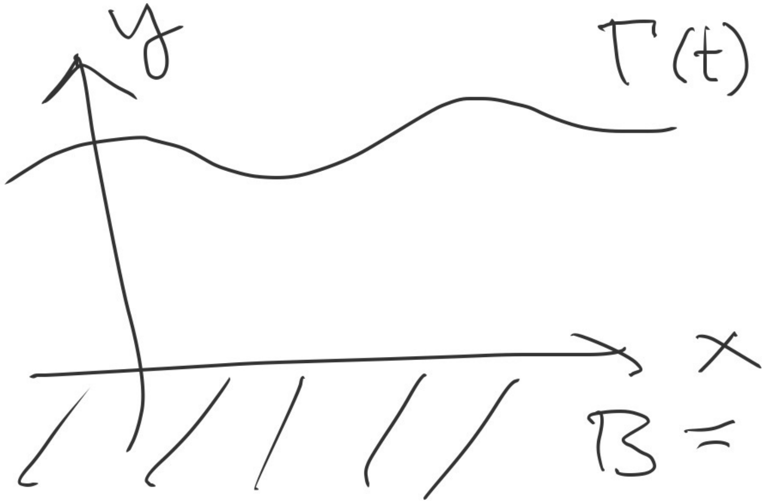
$$\frac{D\vec{V}}{Dt} = -\frac{\nabla p}{\rho} - g\vec{e}_y$$

$$\nabla \cdot \vec{V} = 0 \qquad \rho = 1$$

$$\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}$$

$$\frac{d\vec{x}}{dt} = \vec{V}(\vec{x}(t), t)$$

(Mostly) 2D today



$\Gamma(t) = \{ (x, \eta(x,t)) \}$

or $= \{ (x(\alpha,t), y(\alpha,t)) \}$

$B = \{ (x, -h) \}$

$y(t) = \eta(x(t), t)$

$\dot{y} = \eta_t + \eta_x \dot{x}$

$v_2 = \eta_t + v_1 \eta_x$

Kinematic BCs

$B: \quad 0 = \vec{V} \cdot \vec{n} = v_2$

$\Gamma(t): \quad \eta_t = v_2 - v_1 \eta_x = \vec{V} \cdot \vec{n}$

$\vec{n} = (-\eta_x, 1)$

Dynamic BC

$\Gamma(t) \quad P = -\sigma \kappa \qquad \kappa = \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)_x = \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}$

$\sigma \geq 0$ coeff. of surface tens.

Vorticity

$$\vec{\omega} = \nabla \times \vec{v}$$

Apply $\nabla \times$ to Euler

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{v} \qquad \text{3D}$$

$$\vec{\omega} = (0, 0, \underbrace{v_{2x} - v_{1y}}_{\omega}) \qquad \frac{D\omega}{Dt} = 0 \qquad \text{2D}$$

If $\vec{\omega} = 0$ at $t = 0$, then $\vec{\omega} = 0 \ \forall t$

then the flow is irrotational

Irrot. flow

$$\nabla \times \vec{\omega} = 0 \quad \Longrightarrow \quad \vec{V} = \nabla \phi$$

ϕ = velocity potential

$$0 = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = \Delta \phi$$

$$\frac{\partial \nabla \phi}{\partial t} + (\nabla \phi \cdot \nabla)(\nabla \phi) = -\nabla p - g e_y$$

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g y + p \right) = 0$$

Bernoulli $$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g y + p = C(t)$$

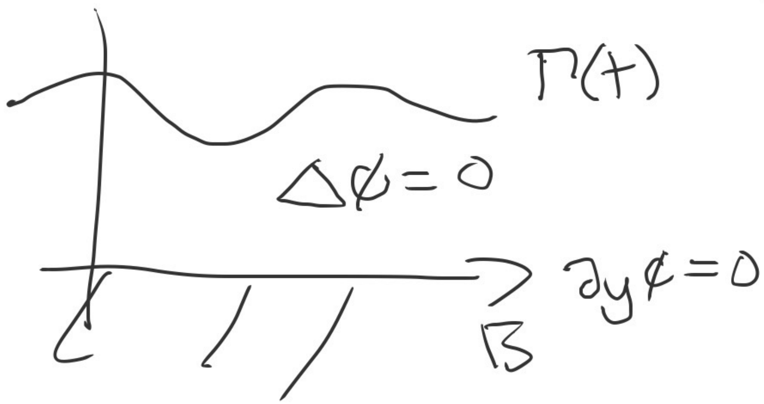
$(= 0)$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gy + p = 0 \qquad \text{at } \Gamma(t)$$

$$p = -\sigma\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x$$

Dynamic BC

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gy - \sigma\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x = 0$$

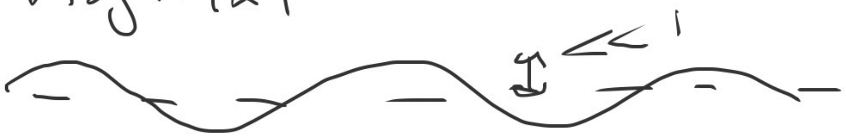


$$\begin{cases} \partial_t \eta = \nabla\phi\cdot\vec{n} & \vec{n} = (-\eta_x, 1) \\ \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}|\nabla\phi|^2 + gy - \sigma\left(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x = 0 \end{cases}$$

Linearisation

$|\eta| \ll 1, \quad |\nabla\phi| \ll 1$

Original



$y = 0$

$\Delta\phi = 0$

$y = -h$

$\phi_y = 0$

Lin.

$\Delta\phi = 0$

$\phi_y = 0$

At Γ

$$\eta_t(x) = \phi_y(x, \eta(x)) - \eta_x(x)\,\phi_x(x, \eta(x)) \approx \phi_y(x, 0)$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + g\eta - \sigma\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = 0$$

$$\leadsto \begin{cases} \eta_t = \phi_y \\ \phi_t = -g\eta + \sigma\eta_{xx} \end{cases}$$

at $y = 0$

$y = 0$

$$\Delta \phi = 0$$

$y = -h$

$$\phi_y = 0$$

$$\begin{cases} \eta_t = \phi_y \\ \phi_t = -g\eta + \sigma \eta_{xx} \end{cases}$$

$' = \frac{\partial}{\partial y} \qquad \dot{} = \frac{\partial}{\partial t}$

$$\phi_k'' - k^2 \phi_k = 0$$

$$\phi_k'(-h) = 0$$

$$\phi_k(y,t) = A_k(t) \cosh(k(y+h))$$

$$\phi = \phi_k(y,t) e^{ikx}$$

$$\eta = \eta_k(t) e^{ikx}$$

$$\begin{cases} \dot{\eta}_k = A_k k \sinh(kh) \\ \dot{A}_k \cosh(kh) = -(g+\sigma k^2) \eta_k \end{cases}$$

$$\ddot{\eta}_k = \dot{A}_k k \sinh(kh)$$

$$= -(g+\sigma k^2) k \tanh(kh) \eta_k$$

$$\eta_k(t) = e^{i\omega_k t}$$

$$\omega_k = \pm \sqrt{(g+\sigma k^2) k \tanh(kh)}$$

$$\eta \sim e^{i(\omega_k t + kx)} = e^{ik(x \pm c_k t)}$$

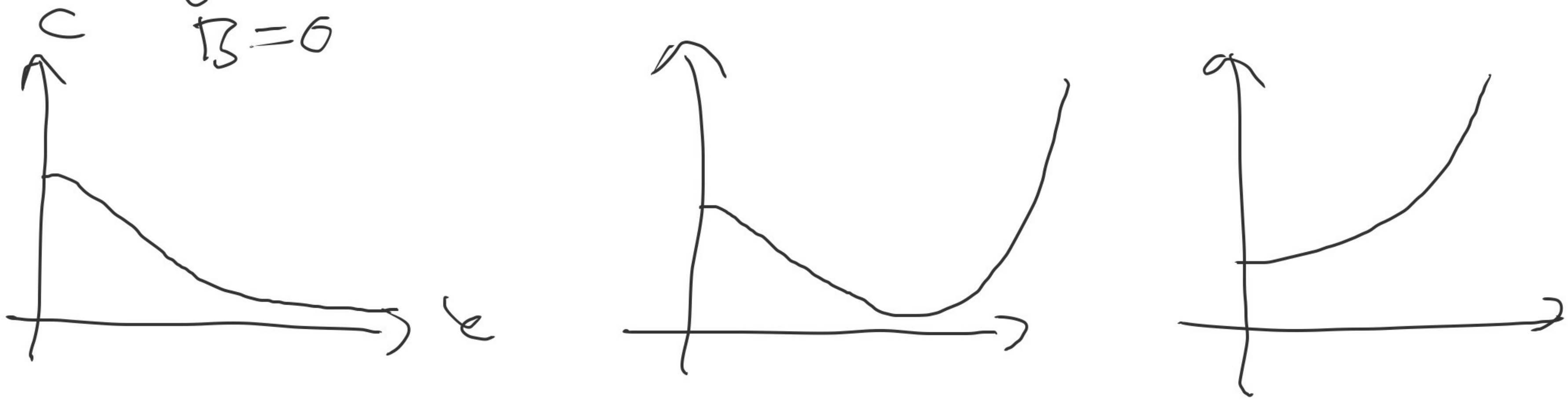
$$c_k = \frac{|\omega_k|}{k}$$

dispersion

$$= \sqrt{\frac{g + \sigma k^2}{k} \tanh(hk)}$$

$$B = \frac{\sigma}{g h^2}$$

$0 < B < \frac{1}{3}$ $\qquad$ $B \geq \frac{1}{3}$



Free boundary ?

Zakharov / Craig-Sulem

$\Phi = \phi(x, \eta(x,t))$

$\nabla\phi \cdot n|_{\Gamma} = G(\eta)\Phi$

DNO

$$\begin{cases} \frac{\partial \eta}{\partial t} - G(\eta)\Phi = 0 \\ \frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2 - \frac{1}{2}\frac{(\Phi_x \eta_x + G(\eta)\Phi)^2}{1+\eta_x^2} + \cdots = 0 \end{cases}$$

Hamiltonian

- Lagrangian formulation

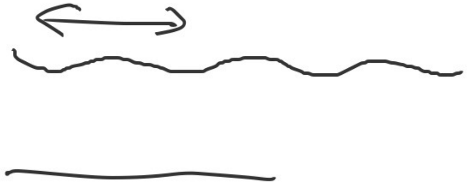
$$\vec{x}(\alpha, t)$$

$$\frac{\partial \vec{X}}{\partial t}(\alpha, t) = \vec{V}(x(\alpha, t), t)$$

$$\frac{\partial^2 \vec{X}}{\partial t^2} = -\nabla p - g\vec{e_y}$$

- Conformal mapping

Mathematical questions



→ Approximations: small-amplitude
shallow water / long waves

wave eq., shallow water eqns
Boussinesq, KdV

→ Travelling waves

Stokes

1880'
~1910 power series
~1970 global bif
1982 verification of Stokes' conj.
Amick, Fraenkel & Toland, Plotnikov

Well-posedness of IVP

70–80's small-amplitude

'97, '99 S. Wu arb. data
finite time

~ '10 Wu Global
German, Masmoudi, Shatah existence

Castro et al wave breaking
Castro –''– self-intersection

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