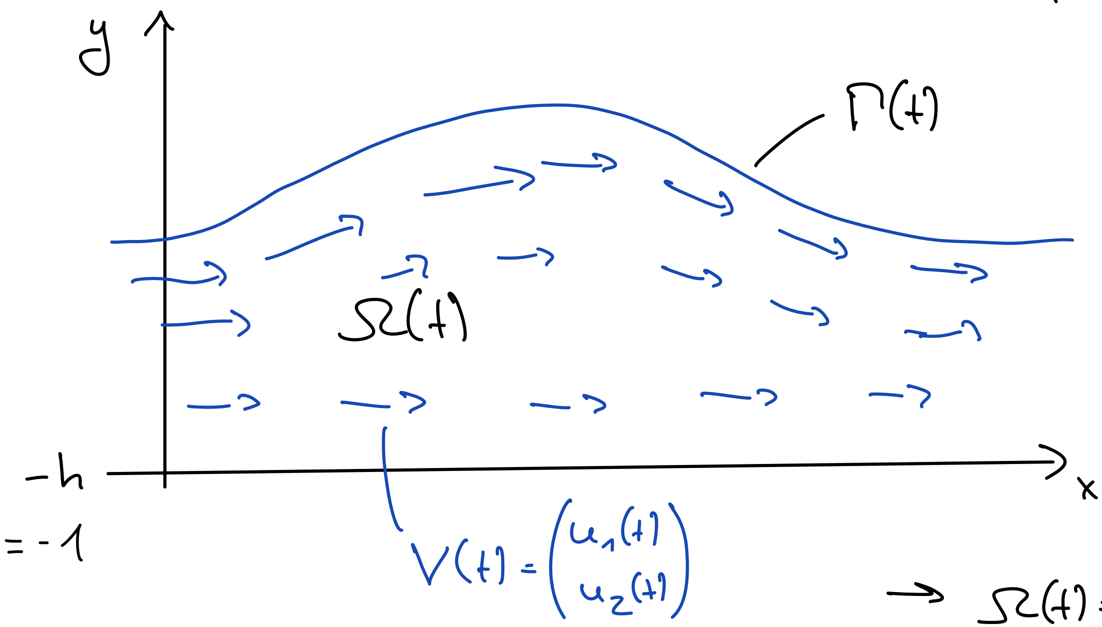
Formal derivation of the KdV eq.

Goal: starting from the water wave problem, derive the Korteweg-de Vries (KdV) equation

$$\partial_t A = \mu \partial_x^3 A + \nu A \partial_x A \quad , \mu, \nu \in \mathbb{R}$$

as an asymptotic model for small, long-wavelength solutions.

1) Repetition: 2d water wave problem



- finite (const.) depth h $(=1)$
- constant density $\rho = 1$
- incompressible and irrotational fluid
  $\nabla \cdot V = 0$ $\quad \nabla \times V = 0$
- surface is a graph $\eta = \eta(x)$

$$\rightarrow \Omega(t) = \{(x,y) : -1 \le y \le \eta(t,x)\}.$$

The evolution of the fluid is given by the Euler equations

$$\begin{cases} \partial_t V + (V \cdot \nabla) V = -\nabla p + \begin{pmatrix} 0 \\ -1 \end{pmatrix} & (g=1) \quad \text{in } \Omega(t) \\ \nabla \cdot V = 0 \end{cases}$$

boundary conditions

•) kinematic: $0 = V \cdot n = u_2$ on $y = -1$

$$\partial_t \eta = u_2 - u_1 \partial_x \eta \quad \text{on } y = \eta(x) \tag{1}$$

•) dynamic: $p = -b\,\kappa$, $\kappa = \partial_x \left(\frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right)$ curvature of $\Gamma(t)$.

irrotational fluid $\Longrightarrow$ $\exists$ velocity potential ϕ s.t. $\nabla \phi = V$.

$$\nabla \cdot V = 0 \Rightarrow \Delta \phi = 0 \quad \text{on } \Omega(t).$$

$\Rightarrow$ Bernoulli: $\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \eta - b\,\kappa = 0$ (2)

Reduction to the surface

$$\begin{cases} \Delta\phi = 0 & \text{on } \Omega(t) \\ \partial_y \phi = 0 & \text{on } y = -1 \end{cases} \Rightarrow \exists K = K(\eta) \text{ s.t. } \partial_y \phi|_\Gamma = K(\eta)\partial_x \phi|_\Gamma$$

$$\Rightarrow u_2|_\Gamma = K(\eta) u_1|_\Gamma .$$

Plugging into (1) and (2) yields

$$\begin{cases} \partial_t \eta = K(\eta) u_1 - u_1 \partial_x \eta \\ \partial_t u_1 = -\frac{1}{2}\partial_x\left(u_1^2 + (K(\eta)u_1)^2\right) - \partial_x \eta + b\,\partial_x^2\left(\frac{\partial_x \eta}{\sqrt{1+(\partial_x\eta)^2}}\right) \end{cases} \text{on } \Gamma(t).$$

Rem: $u_1 \mapsto K(\eta)u_1$ is linear, but $\eta \mapsto K(\eta)u_1$ is highly nonlinear.
$\Rightarrow$ understanding $K(\eta)$ is the main difficulty.

"Lemma"

$$K(\eta)u_1 = -(1+\eta)\partial_x u_1 - \frac{1}{3}\partial_x^3 u_1 + \text{h.o.t.}$$

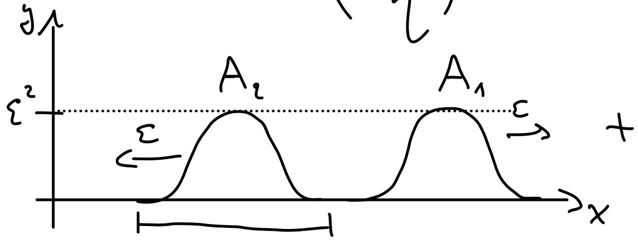
Pf: hopefully later.

Derivation of KdV

Ansatz:

$$\begin{pmatrix} u \\ \eta \end{pmatrix}(t,x) = \varepsilon^2 A_1(\varepsilon(x-t),\varepsilon^3 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^2 A_2(\varepsilon(x+t),\varepsilon^3 t)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$+ \varepsilon^4 B_1(\varepsilon(x-t),\varepsilon^3 t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \varepsilon^4 B_2(\varepsilon(x+t),\varepsilon^3 t)\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$



Rem: In the literature, one often finds the assumption

$$\delta = \frac{h_0}{\lambda} \quad \text{and} \quad \tilde{\varepsilon} = \frac{a}{h_0}$$

(h_0: depth, a: typical amplitude, λ: typical wavelength)

Shallowness $\delta \ll 1$: wavelength much larger than depth

→ long-wave solution

$\tilde{\varepsilon} \ll 1$: small amplitude

For KdV one needs the assumption $\delta^2 = \tilde{\varepsilon}$.

This is "hard-coded" into our ansatz.

Observation:

$$\partial_t u_1 = \varepsilon^5 \partial_\tau (A_1 - A_2) - \varepsilon^3 \partial_\xi (A_1 + A_2) - \varepsilon^5 \partial_\xi (B_1 + B_2) + \mathcal{O}(\varepsilon^7).$$

$$\partial_t \eta = \varepsilon^5 \partial_\tau (A_1 + A_2) + \varepsilon^3 \partial_\xi (-A_1 + A_2)$$

$$\partial_x^j u_1 = \varepsilon^{2+j} \partial_\xi (A_1 - A_2) + \varepsilon^{4+j} \partial_\xi (B_1 - B_2)$$

$$\partial_x^j \eta = \varepsilon^{2+j} \partial_\xi (A_1 + A_2) \quad , \quad j \geq 0.$$

$\tau = \varepsilon^3 t$

$\xi_\pm = \varepsilon (x \pm t).$

Plug into

(i) $\partial_t \eta = K(\eta) u_1 - u_1 \partial_x \eta$

(ii) $\partial_t u_1 = -\frac{1}{2}\left(u_1^2 + (K(\eta) u_1)^2\right) - \partial_x \eta + b \partial_x^2 \left(\frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right)$

(iii) $K(\eta) u_1 = -(1+\eta) \partial_x u_1 - \frac{1}{3} \partial_x^3 u_1 + \text{h.o.t.}$

(iii): $K(\eta) u_1 = -(1 + \varepsilon^2 (A_1 + A_2))\left(\varepsilon^3 \partial_\xi (A_1 - A_2) + \varepsilon^5 \partial_\xi (B_1 - B_2)\right) - \frac{1}{3} \varepsilon^5 \partial_\xi^3 (A_1 - A_2) + \mathcal{O}(\varepsilon^7).$

$$= -\varepsilon^3 \partial_\xi (A_1 - A_2)$$
$$+ \varepsilon^5 \left(-(A_1 + A_2) \partial_\xi (A_1 - A_2) - \partial_\xi (B_1 - B_2) - \frac{1}{3} \partial_\xi^3 (A_1 - A_2) \right) + \mathcal{O}(\varepsilon^7).$$

(i) $$\varepsilon^5 \partial_\tau (A_1 + A_2) + \varepsilon^3 \partial_\xi (-A_1 + A_2)$$

$$= -\varepsilon^3 \partial_\xi (A_1 - A_2) + \varepsilon^5 \Big(-(A_1 + A_2)\partial_\xi (A_1 - A_2) - \partial_\xi (B_1 - B_2) - \frac{1}{3}\partial_\xi^3 (A_1 - A_2) \Big)$$

$$- \varepsilon^5 (A_1 - A_2)\partial_\xi (A_1 + A_2) + \mathcal{O}(\varepsilon^7).$$

$$\Rightarrow \partial_\tau (A_1 + A_2) = -\frac{1}{3}\partial_\xi^3 (A_1 - A_2) - 2A_1 \partial_\xi A_1 + 2A_2 \partial_\xi A_2 - \partial_\xi (B_1 - B_2) + \mathcal{O}(\varepsilon^2).$$

(ii) 1. $$(K(\eta) u_1)^2 = \mathcal{O}(\varepsilon^6)$$

2. $$\partial_x^2 \left(\frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right) = \partial_x^3 \eta + \partial_x^2 \mathcal{O}((\partial_x \eta)^3) = \varepsilon^5 \partial_\xi^3 (A_1 + A_2) + \mathcal{O}(\varepsilon^{11})$$

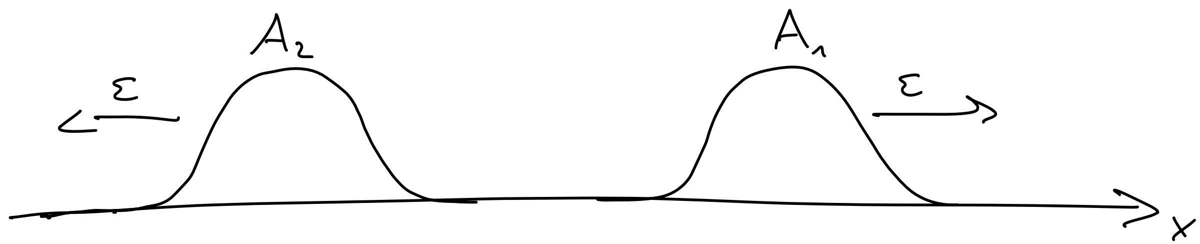
$$\Rightarrow \varepsilon^5 \partial_\tau (A_1 - A_2) - \varepsilon^3 \partial_\xi (A_1 + A_2) - \varepsilon^5 \partial_\xi (B_1 + B_2)$$

$$= -\frac{1}{2}\varepsilon^5 \partial_\xi (A_1 - A_2)^2 - \varepsilon^3 \partial_\xi (A_1 + A_2) + \varepsilon^5 b\, \partial_\xi^3 (A_1 + A_2) + \mathcal{O}(\varepsilon^7).$$

$$\Rightarrow \partial_\tau (A_1 - A_2) = b\, \partial_\xi^3 (A_1 + A_2) - A_1 \partial_\xi A_1 + \partial_\xi (A_1 A_2) - A_2 \partial_\xi A_2 + \partial_\xi B_1 + \partial_\xi B_2$$

$$+ \mathcal{O}(\varepsilon^2).$$

Rem: If A_1, A_2 are sufficiently localized in space, the term $\partial_\xi(A_1 A_2)$ is small over the "typical time interval" $t = \mathcal{O}(\varepsilon^{-3})$ since they only meet for a relatively short timespan $\mathcal{O}(\varepsilon^{-1})$.



$$\partial_\tau(A_1 + A_2) = -\frac{1}{3}\partial_\xi^3(A_1 - A_2) - 2A_1\partial_\xi A_1 + 2A_2\partial_\xi A_2 - \partial_\xi B_1 + \partial_\xi B_2 \qquad (\mathrm{I})$$

$$\partial_\tau(A_1 - A_2) = b\,\partial_\xi^3(A_1 + A_2) - A_1\partial_\xi A_1 - A_2\partial_\xi A_2 + \partial_\xi B_1 + \partial_\xi B_2 \qquad (\mathrm{II})$$

$\boxed{\mathrm{I} + \mathrm{II}:}$

$$2\partial_\tau A_1 = -\left(\frac{1}{3} - b\right)\partial_\xi^3 A_1 + \underline{\left(\frac{1}{3} + b\right)\partial_\xi^3 A_2} - 3A_1\partial_\xi A_1 + \underline{\overbrace{A_2\partial_\xi A_2}^{\frac{1}{2}\partial_\xi A_2^2}} + \underline{2\partial_\xi B_2}$$

$\boxed{\mathrm{I} - \mathrm{II}:}$

$$2\partial_\tau A_2 = \underline{-\left(\frac{1}{3} + b\right)\partial_\xi^3 A_1} + \left(\frac{1}{3} - b\right)\partial_\xi^3 A_2 \underline{- A_1\partial_\xi A_1} + 3A_2\partial_\xi A_2 \underline{- 2\partial_\xi B_1}.$$

Now, set

$$2B_2 + \frac{1}{2}A_2^2 + \left(\frac{1}{3}+b\right)\partial_\xi^2 A_2 = 0$$

$$2B_1 + \frac{1}{2}A_1^2 + \left(\frac{1}{3}+b\right)\partial_\xi^2 A_1 = 0$$

Then, we obtain to leading order two decoupled KdV equations:

$$\boxed{\begin{aligned} \partial_\tau A_1 &= -\left(\frac{1}{6}-\frac{b}{2}\right)\partial_\xi^3 A_1 - \frac{3}{2}A_1\partial_\xi A_1 \\ \partial_\tau A_2 &= \left(\frac{1}{6}-\frac{b}{2}\right)\partial_\xi^3 A_2 + \frac{3}{2}A_2\partial_\xi A_2 . \end{aligned}}$$

Rem: •) The sign of the dispersive term changes at $b=\frac{1}{3}$.

•) For $b=\frac{1}{3}+2\nu\varepsilon^2$, making the ansatz

$$\begin{pmatrix} u_1 \\ \eta \end{pmatrix}(t,x) = \varepsilon^4 A(\varepsilon(x\pm t), \varepsilon^5 t)\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} + \mathcal{O}(\varepsilon^5)$$

one obtains the Kawahara equation

$$\partial_\tau A = \mp\nu\,\partial_\xi^3 A \pm \frac{1}{90}\partial_\xi^5 A \pm \frac{3}{2}A\partial_\xi A \qquad , \quad \tau=\varepsilon^5 t,\ \xi=\varepsilon(x\pm t).$$

instead of the KdV equation.

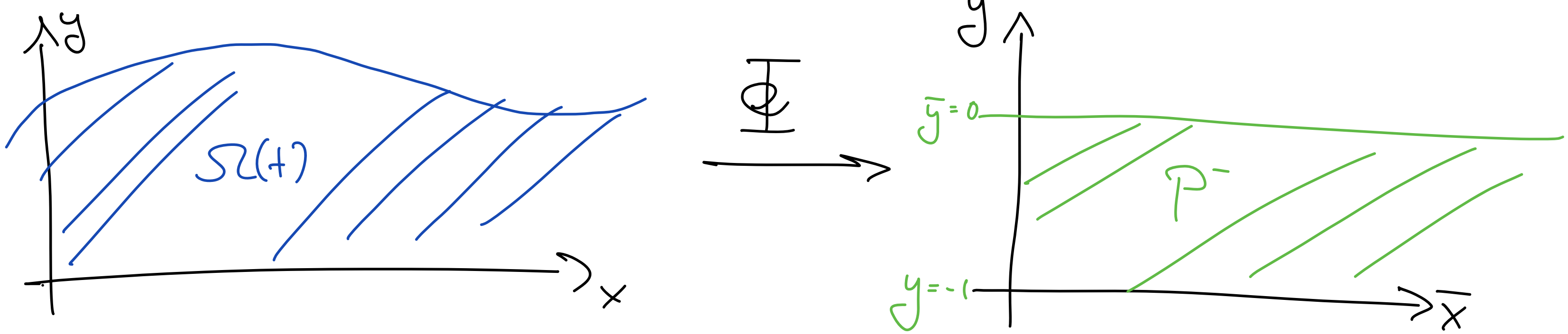
The operator $K(\eta)$ (treatment roughly following Schneider, Wayne '00)

Recall: $\begin{cases} \Delta\phi = 0 & \text{on } \Omega(t) \\ \partial_y \phi = 0 & \text{at } y=-1 \end{cases}$, then $\partial_y \phi\big|_\Gamma = K(\eta)\,\partial_x \phi\big|_\Gamma$.

Idea: map time-dependent domain $\Omega(t)$ to fixed domain

$$P^- = \{(\bar{x},\bar{y}) : -1 \le \bar{y} \le 0\}$$

via a conformal (i.e. holomorphic in $z = x+iy$ and invertible) map $\Phi : \Omega(t) \to P^-$ and solve Laplace problem on P^-.



Lemma: $\exists$ conf. map $\Phi : \Omega \to P^-$ with $\Phi_2(x,-1) = -1$, $\Phi_2(x,\eta(x)) = 0$.
If $\Gamma = \{(x,\eta(x)),\ x\in\mathbb{R}\}$ then

$$h^{-1}(\bar{x}) = \Phi_1^{-1}(\bar{x},\eta(\bar{x})) = \bar{x} + \int_{-\infty}^{\bar{x}} \eta(\xi)\,d\xi - \mathcal{F}^{-1}\left(\frac{k\cosh(k) - \sinh(k)}{ik\sinh(k)}\,\hat{\eta}(k)\right)(\bar{x}).$$

Pf: To construct $\Phi^{-1} : P^- \to \Omega$, solve the Cauchy-Riemann eq:

$$\partial_x v_1 + \partial_y v_2 = 0 \quad , \quad \partial_x v_2 - \partial_y v_1 = 0$$

with $v_2(x,0) = \eta(x)$, $v_2(x,-1) = -1$.

$$\Rightarrow h^{-1}(x) = v_1(x,0) .$$

Note: v_2 satisfies $\begin{cases} \Delta v_2 = 0 & \text{in } P^- \\ v_2(x,0) = \eta(x) \\ v_2(x,-1) = -1 \end{cases}$

Ansatz $v_2 = -y + u(x,y)$ and FT in x yields

$$-k^2 \hat{u} + \partial_y^2 \hat{u} = 0 , \quad \hat{u}(k,-1) = 0 , \quad \hat{u}(k,0) = \hat{\eta}(k)$$

$$\Rightarrow \hat{u} = \hat{C}_k \sinh(k(y+1)) \quad , \quad \hat{C}_k = \frac{\hat{\eta}(k)}{\sinh(k)} .$$

$\Rightarrow$ conjugate function to v_2:

$$v_1(x,y) = x + \int_{-\infty}^{x} \eta(\xi)\, d\xi - \mathcal{F}^{-1}\left(\frac{k \cosh(k(y+1)) - \sinh(k)}{ik \sinh(k)} \hat{\eta}(k) \right)(x,y)$$

Setting $y=0$ yields the statement. $\square$

Cor: $h^{-1}: \mathbb{R} \to \mathbb{R}$ is invertible and

$$\partial_x h^{-1}(x) = 1 + 2\eta(x) - \mathcal{F}^{-1}\left(\underbrace{\frac{k\cosh(k)}{\sinh(k)}}_{=1+\mathcal{O}(k^2)}\hat{\eta}(k)\right) = 1 + \eta(x) + \mathcal{O}(\partial_x^2)\eta.$$

Lemma: $K(\eta)u_1 = K_0[u_1 \circ h^{-1}] \circ h$ with a Fourier multiplier K_0 s.t.

$[K_0 = K(0)]$ $\qquad \left(\widehat{K_0 u_1}\right)(k) = -i\tanh(k)\hat{u}(k) = \left(-ik - \frac{(ik)^3}{3} + \mathcal{O}(|k|^5)\right)\hat{u}(k).$

Pf: $f(x,y) = u_1(x,y) + i u_2(x,y)$ is analytic in $x+iy$ due to incompressibility $(\nabla \cdot V = 0)$ and irrotationality $(\nabla \times V = 0)$.

Let $\Phi : \Omega \to P^-$ be a conf. map. Then $g(\tilde{z}) = f(\Phi^{-1}(\tilde{z}))$, $\tilde{z} = \tilde{x} + i\tilde{y}$ is analytic. Then $\tilde{V} = \begin{pmatrix}\tilde{u}_1 \\ \tilde{u}_2\end{pmatrix} = \begin{pmatrix}\mathrm{Re}(g) \\ \mathrm{Im}(g)\end{pmatrix}$ has a potential $\tilde{\phi}$ with

$$\begin{cases} \Delta\tilde{\phi} = 0 & \text{on } P^- \\ \partial_{\tilde{y}}\tilde{\phi} = 0 & \text{at } y = -1 \end{cases} \implies \tilde{u}_2(\tilde{x}, 0) = \partial_{\tilde{y}}\tilde{\phi}(x,0) = K_0(\partial_{\tilde{x}}\tilde{\phi}(x,0)) = K_0\tilde{u}_1(x,0).$$

The Fourier symbol of K_0 can be found by solving the Laplace problem on P^-. (see Erik's talk or Schneider, Wayne '00, Lem. 3.5). $\square$

Pf of "Lemma":

$$K(\eta)u_1 = \left(K_0[u_1\circ h^{-1}]\right)\circ h$$

We only need to expand $K_0[u_1\circ h^{-1}] = -\partial_x(u_1\circ h^{-1}) - \frac{1}{3}\partial_x^3(u_1\circ h^{-1}) + \mathcal{O}(\partial_x^5)$.

$$-\partial_x(u\circ h^{-1}) = -(\partial_x u)\circ h^{-1}\cdot \partial_x h^{-1} = -(\partial_x u)\circ h^{-1}\cdot\left(1+2\eta - \mathcal{F}^{\sim}\left(\frac{k\cosh(k)}{\sinh(k)}\hat{\eta}(k)\right)\right)(\bar{x})$$

$$= -(\partial_x u)\circ h^{-1}\left(1+\eta\circ h^{-1} + \mathcal{O}(\partial_x^2)\right)$$

using $\frac{k\cosh(k)}{\sin(k)} = 1 + \frac{k^2}{3} + \mathcal{O}(k^4)$.

Recall that $u, \eta \sim \varepsilon^2$, $\partial_x^j \sim \varepsilon \Rightarrow \partial_x u\, \partial_x^2\eta \sim \varepsilon^7$ (everything else is even smaller)

$$\Rightarrow -\partial_x(u\circ h^{-1})\circ h = -(1+\eta)\partial_x u + \text{h.o.t.}$$

Similarly: $\partial_x^3(u\circ h^{-1})\circ h = \underbrace{\partial_x^3 u}_{\varepsilon^5}\cdot \underbrace{\partial_x h^{-1}}_{(1+\varepsilon^2)} + \underbrace{(\partial_x^2 u)}_{\varepsilon^4}\cdot \underbrace{\partial_x(\partial_x h^{-1})^2}_{\varepsilon^3} + \underbrace{\partial_x^2 u}_{\varepsilon^4}\cdot \underbrace{\partial_x h^{-1}}_{(1+\varepsilon^2)}\underbrace{\partial_x^2 h^{-1}}_{\varepsilon^3}$
$+ \underbrace{\partial_x u}_{\varepsilon^3}\, \underbrace{\partial_x^3 h^{-1}}_{\varepsilon^4}$

Using $\partial_x h^{-1} \sim 1+\varepsilon^2$, $\partial_x^2 h^{-1} \sim \varepsilon^3$, $\partial_x^3 h^{-1}\sim\varepsilon^4$ we find

$$\partial_x^3(u\circ h^{-1})\circ h = \partial_x^3 u + \text{h.o.t.}$$

$$\Rightarrow K(\eta)u_1 = -(1+\eta)\partial_x u_1 - \frac{1}{3}\partial_x^3 u_1 + \text{h.o.t.}$$

$\square$

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