Traveling waves of KdV eq.

⇝ Part I ⇝ explicit formula for sol. waves

Part II ⇝ existence of sol. waves with variational.

⇝ 2 weeks ago: Introduction water wave problem

last week KdV as amplitude eq. $\partial_t A = \mu \partial_x^3 A + \nu A \partial_x A$,

$\mu, \nu \in \mathbb{R}$

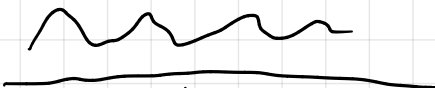
⇝ $\partial_t A = -\partial_x^3 A + 6 A \partial_x A$,

Reminder

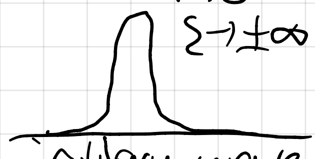
A trav. wave for the KdV eq. is a solution of the form

$A(x,t) = v(\underbrace{x-ct}_{\xi})$, $c \in \mathbb{R}$ is wave speed,

- $c>0$ right-going
- $c<0$ left-going
- $c=0$ steady state.



periodic



solitary wave

Insert $A(x,t) = v(x-ct)$ into $\partial_t A = -\partial_x^3 A + 6A\partial_x A$

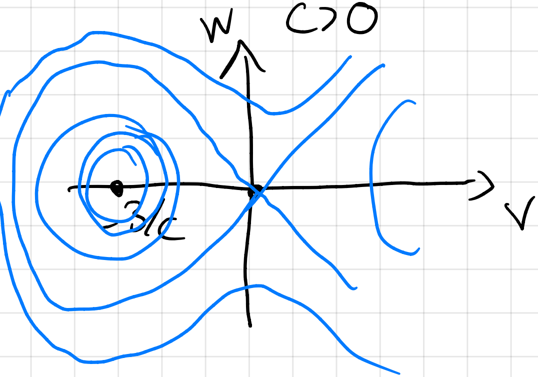
$\rightsquigarrow$ $-cv' = -v''' + 6vv' = -v''' + 3(v^2)'$, $\quad ' \hat{=}$ derivative wrt. ξ

$\rightsquigarrow$ Integration wrt. ξ : $-cv + v'' - 3v^2 \overset{(1)}{=} B_1$, B_1 int. constant.

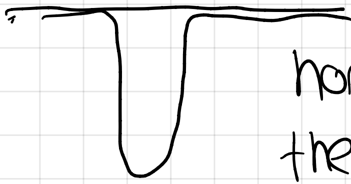
Since $\lim\limits_{|\xi|\to\infty} v(\xi) = \lim\limits_{|\xi|\to\infty} v''(\xi) = 0$ $\rightsquigarrow$ choose $B_1 = 0$.

$\rightsquigarrow$ Write (1) as a first order system

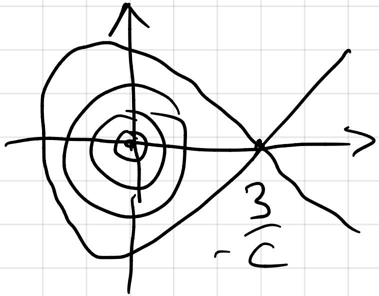
$$v' = \omega$$
$$\omega' = cv + 3v^2$$



$\rightarrow$



homocl. orbit represents the solitary



$\rightsquigarrow \lim\limits_{|\xi|\to\infty} \alpha = -\frac{3}{c} \neq 0$

$\rightsquigarrow$ Focus or $c>0$. Derive explicit.

$-cv + v'' - 3v^2 = 0$

Multiply by v' $\rightsquigarrow -cvv' + v'v'' - 3v^2v' = 0$

$= -\frac{c}{2}(v^2)' + \frac{1}{2}((v')^2)' - (v^3)'$

Integrate $\rightsquigarrow -\frac{c}{2}v^2 + \frac{1}{2}(v')^2 - v^3 = B_2$, B_2 int. constant.

$\hookrightarrow B_2 = 0$

$\rightsquigarrow (v')^2 = v^2(c+2v)$

Separation of variables $\int \frac{1}{v\sqrt{c+2v}}\,dv \overset{(*)}{=} \int 1\,d\xi$

Reminder: $\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$, $\tanh^2(x) + \operatorname{sech}^2(x) \overset{(I)}{=} 1$

$$\frac{d}{dx}\left(\operatorname{sech}(x)\right) = -\tanh(x)\operatorname{sech}(x)$$

$$\frac{d}{dx}\left(\operatorname{sech}^2(x)\right) \overset{(II)}{=} -2\tanh(x)\operatorname{sech}^2(x)$$

Substitution $v = -\frac{1}{2} c \operatorname{sech}^2(s)$, $\dfrac{dv}{ds} \overset{(II)}{=} c \operatorname{sech}^2(s)\tanh(s)$

Insert in ⊛

$$\int \frac{1}{v\sqrt{c+2v}}\, dv \overset{\text{Sub.}}{=} \int \frac{\cancel{c}\, \cancel{\operatorname{sech}^2(s)}\tanh(s)}{-\frac{1}{2}\cancel{c}\,\cancel{\operatorname{sech}^2(s)}\sqrt{c+\cancel{2}\left(-\frac{1}{\cancel{2}} c \operatorname{sech}^2(s)\right)}}\, ds$$

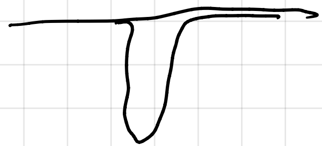
$$= (-2)\cdot \int \frac{\tanh(s)}{\sqrt{c\cdot\underbrace{(1-\operatorname{sech}^2(s))}_{\overset{I}{=}\tanh^2(s)}}}\, ds = (-2)\int \frac{\cancel{\tanh(s)}}{\sqrt{c}\,\cancel{\tanh(s)}}\, ds$$

$$= \left(-\frac{2}{\sqrt{c}}\right) s \overset{(*)}{=} \xi + B_3 \,, \quad B_3 \text{ int. constant.}$$

$$\rightsquigarrow \; s = -\frac{\sqrt{c}}{2}\left(\xi + B_3\right)$$

$$\rightsquigarrow \; v(\xi) \overset{\text{Resub.}}{=} -\frac{1}{2} c \cdot \operatorname{sech}^2\left(-\frac{\sqrt{c}}{2}\left(\xi + B_3\right)\right) = -\frac{1}{2} c \operatorname{sech}^2\left(\frac{\sqrt{c}}{2}\left(\xi + B_3\right)\right)$$

Remark.



- This solution corresponds to the 1-soliton
- explicit formula for 2 soliton

$$A(x,t) = -12 \, \frac{3 + 4\cosh(2\xi + 24t) + \cosh(4\xi)}{\left(3\cosh(\xi - 12t) + \cosh(3\xi + 12t)\right)^2} \,, \quad \xi = x - 16t$$

One can show $A(x,t) \sim -8\,\mathrm{sech}^2(2\xi \mp \frac{1}{2}\log 3) - 2\,\mathrm{sech}^2(\eta \pm \frac{1}{2}\log 3)$

for $t \to \pm\infty$ $\qquad \eta = x - 4t$

Part II : existence of solitary waves with variational methods

In general $\quad A_t + (A^2 + LA)_x = 0 \quad , \quad \widehat{LA}(t,x) = m(k)\hat{A}(t,k)$

(KdV) $\quad A_t + (A^2 - \partial_x^2 A)_x = 0 \qquad m(k) = k^2$

$\mu = 1, \ \nu = -1$

$A(t,x) = v(\underbrace{x-ct}_{\xi})$ solitary wave sol. $\rightsquigarrow$ $-cv + v^2 - \partial_\xi^2 v = 0$ (3)

Define functional

$$M(v) = \frac{1}{2}\int_{\mathbb{R}} v^2 \, d\xi \quad , \quad E(v) = \int_{\mathbb{R}} \frac{1}{2}(\partial_\xi v)^2 + \frac{1}{3} v^3 \, d\xi$$

$\rightsquigarrow \inf \{ E(v) \mid v \in H^1(\mathbb{R}) \setminus \{0\} \text{ s.t. } M(v) = q \} , \ q > 0$ fixed

Notice:

If v satisfies the minimization problem, then $E'(v) = \lambda M'(v)$ for some $\lambda \in \mathbb{R}$

$$E'(v)w = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(E(v+\varepsilon w) - E(v)\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\int_{\mathbb{R}} \frac{1}{2}\left(\partial_\xi(v+\varepsilon w)\right)^2 + \frac{1}{3}(v+\varepsilon w)^3 \, d\xi - \int_{\mathbb{R}} \frac{1}{2}(\partial_\xi v)^2 + \frac{1}{3} v^3 d\xi\right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}}\Bigg(\int_{\mathbb{R}} \frac{1}{2}\left(\cancel{(\partial_\xi v)^2} + 2\cancel{\varepsilon}\, \partial_\xi v\, \partial_\xi w\right) + \underbrace{\varepsilon^{\cancel{2}} (\partial_\xi w)^2}_{\to 0}$$

$$+ \frac{1}{3}\left(\cancel{v^3} + 3\cancel{\varepsilon} v^2 w + \underbrace{3\varepsilon^{\cancel{2}} v w^2 + \varepsilon^{\cancel{3}} w^3}_{\to 0} d\xi\right.$$

$$- \int_{\mathbb{R}} \frac{1}{2}\cancel{(\partial_\xi v)^2} + \cancel{\frac{1}{3} v^3} d\xi$$

$$= \int_{\mathbb{R}} (\partial_\xi v)(\partial_\xi w) + v^2 w \, d\xi \overset{\text{part. int.}}{=} \int \left(-\partial_\xi^2 v + v^2\right) w \, d\xi$$

$$M'(v)w = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(M(v+\varepsilon w) - M(v)\Big) = \ldots = \int_{\mathbb{R}} \underline{v}\, w \, d\xi$$

$\rightsquigarrow$ $E'(v) = \lambda M'(v)$ $\rightsquigarrow$ $-\partial_\xi^2 v + v^2 = \lambda v$

$\Leftrightarrow$ v solves (3) with $\lambda = c$

$\Leftrightarrow$ v is a solitary-wave solution and the wave speed is λ.

Task: Find minimizers of the functional E