Stability of KdV solitary waves

KdV: $u_t + u_{xxx} - \frac{1}{2}(u^2)_x = 0$

Last time: solitary waves $u(t,x) = v(x-ct)$, $v = v^c$, $c>0$

Stability !?

Observations:

1) Energy: $E(u) = \int \left(\frac{1}{2}u_x^2 + \frac{1}{6}u^3\right)dx$, $u \in X := H^1(\mathbb{R})$
KdV globally well-posed

$E'(u) \equiv -u_{xx} + \frac{1}{2}u^2$, $E''(u) \equiv -\frac{\partial^2}{\partial x^2} + u$

$J := \frac{\partial}{\partial x}$ $\rightarrow$ KdV: $u_t = J E'(u)$
skew!

energy conserved: $\frac{d}{dt} E(u) = \langle E'(u), u_t \rangle = \langle E'(u), J E'(u) \rangle \overset{J \text{ skew}}{=} 0$

2) Translational invariance: Unitary group $T(s)$ acting on X leaves eq. invariant; $T(s)u = u(\cdot - s)$ ($\rightsquigarrow$ traveling solutions); $T'(0) = -\frac{\partial}{\partial x}$, skew
solitary wave: $u(t,\cdot) = T(ct)v$

Induces another conserved quantity $Q(u) = \frac{1}{2}\langle Bu, u\rangle$, where $B = B^*$, $JB \supset T'(0)$ (KdV: $B \equiv -1$)

KdV: $Q(u) = -\frac{1}{2}\int u^2 dx$, $Q'(u) \equiv -u$, $Q''(u) \equiv -1$

$\frac{d}{dt} Q(u) = \langle Bu, JE'(u)\rangle = -\langle T'(0)u, E'(u)\rangle = 0$
E invariant under T

3) Have in mind a constrained minimisation problem:

$$\min_{u \in X} E(u) \quad \text{s.t.} \quad Q(u) = \underbrace{q_0}_{\text{const.}}$$

first order condition: $E'(u) - c\,Q'(u) = 0$, satisfied for $u = v$

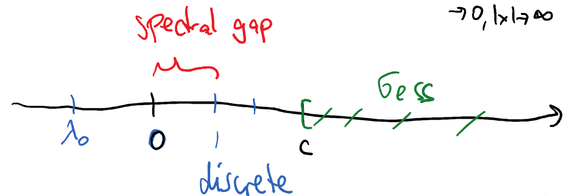
↑ Lagrange multiplier

$(-v_{xx} + \frac{1}{2}v^2 + cv = 0)$

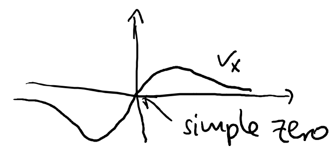
concept: stability ⟺ local minimum ⟺ second order condition

So study $H := E''(v) - c\,Q''(v) \equiv -\frac{\partial^2}{\partial x^2} + \underbrace{v}_{\to 0,\ |x|\to\infty} + c$, positive definite !?

spectrum:



a) 0 eigenvalue, ker $H_{KdV} = \langle \underbrace{v_x}_{-T'(0)v} \rangle$

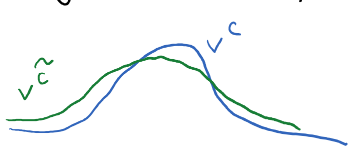


b) Sturm-Liouville theory ⇒ exactly one negative, simple eigenvalue λ_0

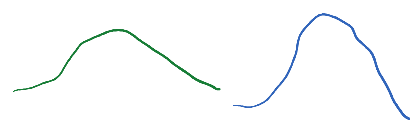
problem!

What to do with them?

a) What is a good notion for "stability"?



$\|v^c - v^{\tilde c}\|$ small if $|c - \tilde c|$ small, but after $t \gtrsim \frac{1}{|c-\tilde c|}$ big difference:



→ look only at difference in shape, independent of position

→ "orbital stability": v-orbit $\mathcal{O} := \{T(ct)v : t \in \mathbb{R}\}$ stable

:⟺ $\forall \varepsilon > 0\ \exists \delta > 0 : \|u_0 - v\| < \delta \Rightarrow \underbrace{\inf_{s\in\mathbb{R}} \|u(t) - T(s)v\|}_{= \tilde d(u(t), \mathcal{O})} < \varepsilon$, $t \geq 0$

(u solves eq., $u(0) = u_0$)

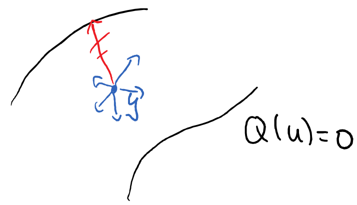
b) Convexity of $d(c) := E(v) - c\,Q(v)$

principle: $d''(c) > 0$ → stability, $d''(c) < 0$ → instability

$d'(c) = \langle \underbrace{E'(v) - c\,Q'(v)}_{=0}, v_c \rangle - Q(v) = -Q(v)$

$$d''(c) = -\langle Q'(v), v_c\rangle \underset{\substack{\uparrow \\ 0 = (E'(v) - cQ'(v))_c = Hv_c - Q'(v)}}{=} -\langle Hv_c, v_c\rangle$$

if $d''(c)>0$: v_c seemingly unstable direction of H, but not important!



KdV: $d'(c) = -Q(v) = \frac{1}{2}\int v^2\,dx \simeq c^2 \int \operatorname{sech}^4(\sqrt{c}\,x)\,dx \simeq c^{3/2}$

$\Rightarrow d''(c)>0$

More precisely:

Prop: Assume $\langle Q'(v), y\rangle = 0$, $(T'(0)v, y)_X = 0$, $y \neq 0$.

Then $\langle Hy, y\rangle > 0$ $\quad (\rightsquigarrow \langle Hy,y\rangle \gtrsim \|y\|^2)$.

Proof:
$$X = \underbrace{\langle \chi \rangle}_{\text{eigenvector to } \lambda_0<0} \oplus \underbrace{\langle T'(0)v\rangle}_{=\ker H = \langle v_x\rangle} \oplus \underbrace{P}_{H \geq \delta > 0 \text{ (spectral gap!)}}$$

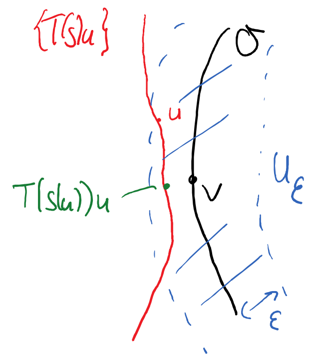
$$y = a\chi + p$$

$$v_c = a_0\chi + b_0 T'(0)v + p_0 \Rightarrow 0 > \langle Hv_c, v_c\rangle = a_0^2\lambda_0 + \langle Hp_0, p_0\rangle$$

$$0 = \langle Q'(v), y\rangle = \langle Hv_c, y\rangle = a a_0 \lambda_0 + \langle Hp_0, p\rangle$$

$$\Rightarrow \langle Hy,y\rangle = a^2\lambda_0 + \langle Hp,p\rangle \underset{\substack{\text{CSU} \\ \text{on } P}}{\geq} a^2\lambda_0 + \frac{\langle Hp_0,p\rangle^2}{\langle Hp_0,p_0\rangle} > a^2\lambda_0 - \frac{a^2 a_0^2 \lambda_0^2}{a_0^2\lambda_0} = 0 \,. \ \square$$

Thm: $\exists K, \varepsilon > 0$: $E(u) - E(v) \geq K\|T(s(u))u - v\|^2$, for $u \in U_\varepsilon$, $Q(u) = Q(v)$



$$\rightsquigarrow \ (T(s(u))u, T'(0)v) = 0$$

($\Rightarrow$ orbital stability)

Proof: $w := T(s(u))u - v = a\underbrace{q}_{=Q'(v)} + y \quad , \quad (q,y) = 0$

$$Q(v) = Q(u) = Q\big(T(s(u))u\big) = Q(v) + \underbrace{\langle Q'(v), w\rangle}_{=a\|q\|^2} + O(\|w\|^2) \Rightarrow a = O(\|w\|^2)$$

$$E(u) - E(v) = \Big(E(T(s(u))) - c\,Q\big(T(s(u))\big)\Big) - \big(E(v) - c\,Q(v)\big)$$

$$\underset{E'(v)-cQ'(v)=0}{=} \frac{1}{2}\langle Hw, w\rangle + o(\|w\|^2) \underset{y=w-aq}{=} \frac{1}{2}\langle Hy, y\rangle + o(\|w\|^2)$$

Also, $\langle Q'(v), y\rangle = (q,y) = 0$, $(y, T'(0)v) = (T(s(u))u - v - aq, T'(0)v) = 0$.

$$\overset{\text{Prop.}}{\Rightarrow} \ E(u) - E(v) \geq \tilde{K}\|y\|^2 + o(\|w\|^2) \overset{\|y\| \geq \|w\| - O(\|w\|^2)}{\Rightarrow} E(u) - E(v) \geq K\|w\|^2, \ \|w\| \text{ small.} \ \square$$