APPROXIMATION THEOREMS FOR THE WATER WAVE PROBLEM IN THE ARC LENGTH FORMULATION

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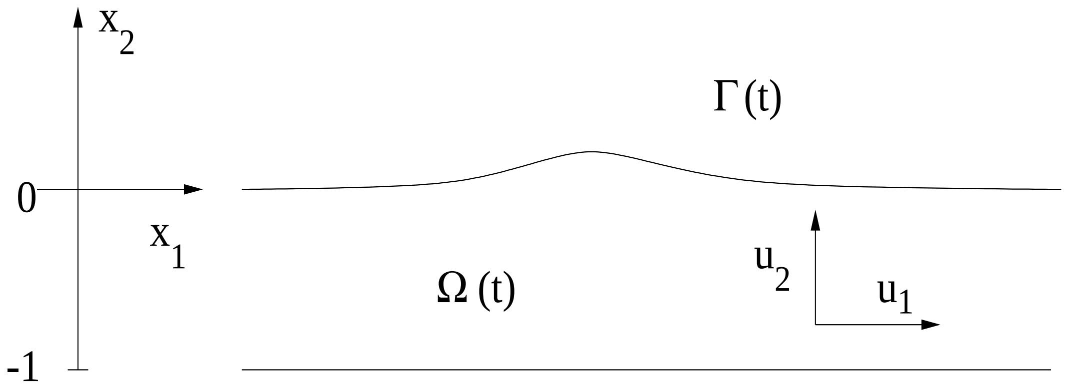
General context

- Justification of approximation equations for pattern forming systems and for water waves
- A typical example: approximation of the 2–D water wave problem by the Korteweg–de Vries (KdV) equation
- Application: prediction of the qualitative behavior of solutions to the original equations with the help of the approximation equations

1. The 2–d water wave problem and the KdV approximation in Eulerian coordinates



- *Law of motion* for the velocity field $V = (u_1, u_2)$ of an incompressible, inviscid fluid in an infinitely long canal of finite depth under the influence of gravity:

$$V_t + (V \cdot \nabla)V = -\nabla p - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{in } \Omega(t), \tag{1}$$

$$\nabla \cdot V = 0 \qquad \text{in } \Omega(t) \tag{2}$$

(incompressible Euler equations)

- *Boundary conditions:*

1. No slip condition on the free top surface $\Gamma(t) = \eta(x_1, t)$, i.e., surface particles remain surface particles:

$$\eta_t = V \cdot \begin{pmatrix} -\eta_{x_1} \\ 1 \end{pmatrix} \qquad \text{at } \Gamma(t), \tag{3}$$

2. Laplace–Young condition for the pressure p:

$$p = -b\kappa \qquad \text{at } \Gamma(t), \tag{4}$$

b : Bond number (proportional to the strength of the surface tension),
κ : curvature,

3. Impermeable bottom B:

$$u_2 = 0 \qquad \text{at } B. \tag{5}$$

- From now on we additionally assume

$$\nabla \times V = 0 \qquad \text{in } \Omega(t), \tag{6}$$

- Then there exists a harmonic velocity potential ϕ and an operator $\mathscr{K} = \mathscr{K}(\eta)$ s.t.

$$V = \nabla\phi \qquad \text{and} \qquad \phi_y = \mathscr{K}\phi_x, \tag{7}$$

where $x = x_1$, $y = x_2$.

- Using (7), the system (1)–(6) can be reduced to

$$\eta_t = \mathscr{K} u_1 - u_1 \eta_x \qquad \text{at } \Gamma(t), \tag{8}$$

$$(u_1)_t = -\eta_x - \tfrac{1}{2}((u_1)^2 + (\mathscr{K} u_1)^2)_x + b\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_{xx} \qquad \text{at } \Gamma(t). \tag{9}$$

- Inserting the long–wave ansatz

$$\begin{pmatrix} \eta \\ u_1 \end{pmatrix}(x,t) = \varepsilon^2 A\left(\varepsilon(x \pm t), \varepsilon^3 t\right)\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} + \mathscr{O}(\varepsilon^3) \qquad (\varepsilon \ll 1)$$

in (8)–(9) yields at leading order in ε the KdV equation

$$A_\tau = \pm\left(\tfrac{1}{6} - \tfrac{b}{2}\right)A_{\xi\xi\xi} \pm \tfrac{3}{2}AA_\xi \tag{10}$$

with $\tau = \varepsilon^3 t, \xi = \varepsilon(x \pm t)$.

- For $b = \frac{1}{3} + 2\nu\varepsilon^2$ one gets, by making the ansatz

$$\begin{pmatrix} \eta \\ u_1 \end{pmatrix}(x,t) = \varepsilon^4 A\left(\varepsilon(x \pm t), \varepsilon^5 t\right)\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} + \mathscr{O}(\varepsilon^5)$$

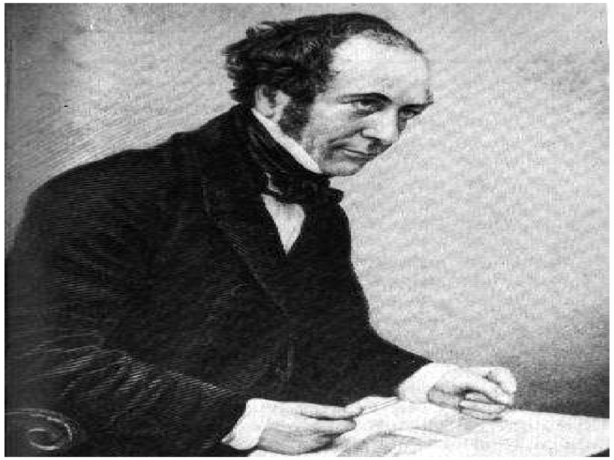
the Kawahara equation

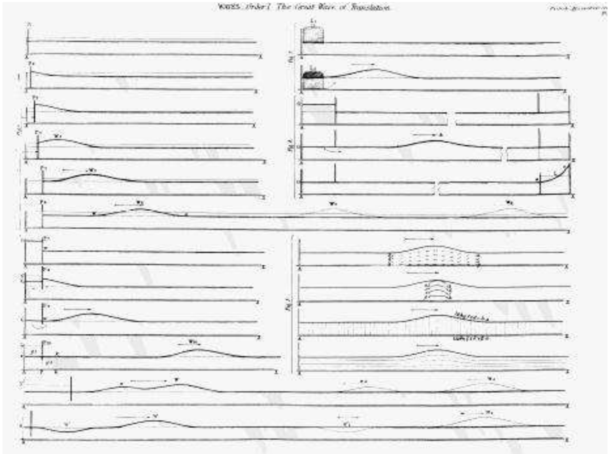
$$\partial_\tau A = \mp \nu \partial_\xi^3 A \pm \tfrac{1}{90}\partial_\xi^5 A \pm \tfrac{3}{2}A\partial_\xi A \tag{11}$$

with $\tau = \varepsilon^5 t, \xi = \varepsilon(x \pm t)$.

- Consequently, the soliton dynamics of the KdV equation and the dynamics of the Kawahara equation are at least approximately present in the 2-d water wave problem.

- Solitons were first observed experimentally by John Scott Russell in 1834 (J. S. Russell: Report on waves. Rep. 14th Meet. Brit. Assoc. Adv. Sci., York, London, John Murray, (1844), 311–390).





- Consequence of solitary waves: speed limits for high speed ferries

(a) HSC operating at sub-critical speed in the Marlborough Sounds

(b) HSC operating near super-critical speed in the Marlborough Sound

- Rigorous justification of the KdV and the Kawahara approximation by proving that the relative error of the approximation is small on the characteristic time scale of the approximation equation.
- Previous approximation proofs on the right time scales:
  Craig (1985), Schneider–Wayne (2000, 2002) : using Lagrangian coordinates
  Bona–Colin–Lannes (2005), Iguchi (2007): using Eulerian coordinates
- We present a new approximation proof using the arc length formulation of the water wave problem.
- The arc length formulation of the water wave problem was introduced by Ambrose–Masmoudi (2005).
- We prove the following theorems:

Theorem 1.1:

For all $b_0, C_0, \tau_0 > 0$ there exist an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leq \varepsilon_0$ and all $b \in \mathbb{R} \setminus \{\frac{1}{3}\}$ with $0 \leq b \leq b_0$ the following is true. Let

$$\eta|_{t=0}(x) = \varepsilon^2 \Phi_1(\varepsilon x), \qquad u_1|_{t=0}(x) = \varepsilon^2 \Phi_2(\varepsilon x)$$

with $\|(\Phi_1, \Phi_2)\|_{H_\xi^{s+8} \cap H_\xi^{s+3}(k)} \leq C_0 \varepsilon^l$, where $\xi = \varepsilon x$, $s \geq 7$, $k > 1$ and $l \geq 0$. Let

$$(A_1)_\tau = \left(\frac{b}{2} - \frac{1}{6}\right)(A_1)_{\xi\xi\xi} - \frac{3}{2} A_1 (A_1)_\xi, \qquad (A_2)_\tau = \left(\frac{1}{6} - \frac{b}{2}\right)(A_2)_{\xi\xi\xi} + \frac{3}{2} A_2 (A_2)_\xi,$$

$$A_1|_{\tau=0} = \frac{1}{2}(\Phi_1 + \Phi_2), \qquad A_2|_{\tau=0} = \frac{1}{2}(\Phi_1 - \Phi_2).$$

Then there exists a unique solution of the 2–D water wave problem (8)–(9) with the above initial conditions satisfying

$$\sup_{t \in [0, \tau_0/\varepsilon^3]} \left\| \begin{pmatrix} \eta \\ u_1 \end{pmatrix} (\cdot, t) - \psi(\cdot, t) \right\|_{H_\xi^s \times H_\xi^{s-1/2}} \lesssim \varepsilon^{4+l}$$

where

$$\psi(x,t) = \varepsilon^2 A_1\left(\varepsilon(x-t), \varepsilon^3 t\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^2 A_2\left(\varepsilon(x+t), \varepsilon^3 t\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Theorem 1.2:

Let $b = \frac{1}{3} + 2\nu\varepsilon^2$. For all $C_0, \tau_0 > 0$ there exist an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leq \varepsilon_0$ the following is true. Let

$$\eta|_{t=0}(x) = \varepsilon^4 \Phi_1(\varepsilon x), \qquad u_1|_{t=0}(x) = \varepsilon^4 \Phi_2(\varepsilon x)$$

with $\|(\Phi_1, \Phi_2)\|_{H_\xi^{s+10} \cap H_\xi^{s+3}(k)} \leq C_0 \varepsilon^l$, where $\xi = \varepsilon x$, $s \geq 7$, $k > 1$ and $l \geq 0$. Let

$$(A_1)_\tau = \nu \partial_\xi^3 A_1 - \frac{1}{90}\partial_\xi^5 A_1 - \frac{3}{2} A_1 \partial_\xi A_1, \qquad (A_2)_\tau = -\nu \partial_\xi^3 A_2 + \frac{1}{90}\partial_\xi^5 A_2 + \frac{3}{2} A_2 \partial_\xi A_2,$$

$$A_1|_{\tau=0} = \frac{1}{2}(\Phi_1 + \Phi_2), \qquad A_2|_{\tau=0} = \frac{1}{2}(\Phi_1 - \Phi_2).$$

Let $[0, \tau_1]$ be the existence interval of A_1, A_2 in $H_\xi^{s+10} \cap H_\xi^{s+3}(k)$ and $\tau_2 = \min\{\tau_0; \tau_1\}$. Then there exists a unique solution of the 2–D water wave problem (8)–(9) with the above initial conditions satisfying

$$\sup_{t \in [0, \tau_2/\varepsilon^5]} \left\| \begin{pmatrix} \eta \\ u_1 \end{pmatrix} (\cdot, t) - \psi(\cdot, t) \right\|_{H_\xi^s \times H_\xi^{s-1/2}} \lesssim \varepsilon^{6+l}$$

where

$$\psi(x,t) = \varepsilon^4 A_1\left(\varepsilon(x-t), \varepsilon^5 t\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^4 A_2\left(\varepsilon(x+t), \varepsilon^5 t\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- Main advantages of the use of the arc length formulation:
  - Proof of the local–wellposedness (Ambrose–Masmoudi) is more elementary and less complex than in Eulerian or in Lagrangian coordinates.
  - Better regularity properties
  - Our error estimates for the KdV approximation are the only ones being uniform w.r.t. the strength of the surface tension as b and ε go to 0.
  - Therefore, the cases with and without surface tension can be handled together in one approximation proof.
  - Optimal powers of ε in our bounds for the error and its spatial derivatives.
  - Arc length parametrization in the KdV– and the Kawahara–regime is close to Eulerian coordinates.
  - More accessible to generalizations

2. The 2–d water wave problem and the KdV approximation in the arc length formulation

- Let $P(t) : \mathbb{R} \to \Gamma(t)$, $\alpha \mapsto P(\alpha,t) = (x(\alpha,t), y(\alpha,t))$ be a parametrization of the top surface by arc length.
- Then we have:

$$(x,y)_t = U\hat{n} + T\hat{t}, \tag{12}$$

where U and T are the normal and the tangential velocity of Γ w.r.t. P.

- T is determined (up to a constant) by the arc length condition, which yields

$$T = \int \theta_\alpha U\,, \tag{13}$$

where $\theta = \arctan(y_\alpha / x_\alpha)$ is the tangent angle.

- The irrotationality of the flow and the Neumann boundary condition (5) imply that for known $\Gamma(t)$ the normal velocity $U(t)$ is uniquely determined by the Lagrangian tangential velocity $v(t)$ via the so–called Birkhoff–Rott integral W.

- The Birkhoff–Rott integral $W : (\gamma, \sigma) \mapsto W(\gamma, \sigma) = W_1(\gamma) + W_2(\sigma)$ is defined by

$$(\operatorname{Re} W_1(\gamma) - i \operatorname{Im} W_1(\gamma))(\alpha, t) = \frac{1}{2\pi i} \mathrm{PV} \int_{-\infty}^{\infty} \frac{\gamma(\alpha', t)}{z(\alpha, t) - z(\alpha', t)} d\alpha', \tag{14}$$

$$(\operatorname{Re} W_2(\sigma) - i \operatorname{Im} W_2(\sigma))(\alpha, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma(\alpha', t)}{z(\alpha, t) - z_B(\alpha')} d\alpha', \tag{15}$$

  where z and z_B are complex representations of arc length parametrizations of the top surface and the bottom. γ is the so–called vortex sheet strength and σ is the so–called source strength.

- σ is uniquely determined by γ via

$$\sigma(\alpha', t) = \frac{1}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} \frac{\gamma(\alpha'', t)}{z(\alpha, t) - z_B(\alpha')} d\alpha''. \tag{16}$$

- γ is uniquely determined by v via

$$\tfrac{1}{2}\gamma + W(\gamma) \cdot \hat{t} = v. \tag{17}$$

- Finally, we have

$$U = W \cdot \hat{n}. \tag{18}$$

- v is governed by the incompressible Euler equations (1)–(2) and the boundary conditions (4)–(5). This yields

$$v_t = -y_\alpha + b\theta_{\alpha\alpha} - (v-T)(v-T)_\alpha + (W\cdot\hat{n})\theta_t\,. \tag{19}$$

- Finally, the 2–D water wave model is equivalent to the following evolutionary system:

$$y_t = (W\cdot\hat{n})cos\theta + Ty_\alpha\,, \tag{20}$$

$$v_t = -y_\alpha + b\theta_{\alpha\alpha} - \delta\delta_\alpha + (W\cdot\hat{n})\theta_t\,, \tag{21}$$

$$\theta_t = W_\alpha\cdot\hat{n} + \tfrac{\gamma}{2}\theta_\alpha - \delta\theta_\alpha\,, \tag{22}$$

$$\delta_{\alpha t} = -(1+c)\theta_\alpha + b\theta_{\alpha\alpha\alpha} - (\delta\delta_\alpha)_\alpha + \left(W_\alpha\cdot\hat{n} + \tfrac{\gamma}{2}\theta_\alpha\right)^2, \tag{23}$$

where

$$\delta = v - T\,, \tag{24}$$

$$c = W_t\cdot\hat{n} + \delta(W_\alpha\cdot\hat{n}) + \tfrac{\gamma}{2}\theta_t + \tfrac{\gamma}{2}\delta\theta_\alpha + (cos\theta - 1)\,. \tag{25}$$

- We will use the equations for y and v to perform the approximation and the equations for θ and δ_α to estimate the derivatives of the error.

- Now, we go to the KdV–scaling:

$$\alpha = \varepsilon^{-1}\underline{\alpha}, \quad y(\alpha,t) = \varepsilon^2 \tilde{y}(\underline{\alpha},t), \quad v(\alpha,t) = \varepsilon^2 \tilde{v}(\underline{\alpha},t)$$

and therefore

$$x(\alpha,t) = \alpha + \varepsilon^5 \tilde{x}(\underline{\alpha},t), \quad z(\alpha,t) = \alpha + \varepsilon^2 \tilde{z}(\underline{\alpha},t), \quad \theta(\alpha,t) = \varepsilon^3 \tilde{\theta}(\underline{\alpha},t),$$

$$\gamma(\alpha,t) = \varepsilon^2 \tilde{\gamma}(\underline{\alpha},t), \quad W(\alpha,t) = \varepsilon^2 \tilde{W}(\underline{\alpha},t), \quad \delta(\alpha,t) = \varepsilon^2 \tilde{\delta}(\underline{\alpha},t),$$

$$U(\alpha,t) = \varepsilon^2 \tilde{U}(\underline{\alpha},t), \quad T(\alpha,t) = \varepsilon^5 \tilde{T}(\underline{\alpha},t).$$

- In this scaling the system becomes

$$\tilde{y}_t = (\tilde{W}\cdot\hat{n})(1 + (cos(\varepsilon^3\tilde{\theta}) - 1)) + \varepsilon^6 \tilde{T}\tilde{y}_{\underline{\alpha}}\,, \tag{26}$$

$$\tilde{v}_t = -\varepsilon\tilde{y}_{\underline{\alpha}} + \varepsilon^3 b\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}} - \varepsilon^3\tilde{\delta}\tilde{\delta}_{\underline{\alpha}} + \varepsilon^3(\tilde{W}\cdot\hat{n})\tilde{\theta}_t\,, \tag{27}$$

$$\tilde{\theta}_t = \tilde{W}_{\underline{\alpha}}\cdot\hat{n} + \varepsilon^3\tfrac{\tilde{\gamma}}{2}\tilde{\theta}_{\underline{\alpha}} - \varepsilon^3\tilde{\delta}\tilde{\theta}_{\underline{\alpha}}\,, \tag{28}$$

$$\tilde{\delta}_{\underline{\alpha}t} = -\varepsilon(1+\varepsilon^2\tilde{c})\tilde{\theta}_{\underline{\alpha}} + \varepsilon^3 b\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}} - \varepsilon^3(\tilde{\delta}\tilde{\delta}_{\underline{\alpha}})_{\underline{\alpha}} + \varepsilon^3\left(\tilde{W}_{\underline{\alpha}}\cdot\hat{n} + \varepsilon^3\tfrac{\tilde{\gamma}}{2}\tilde{\theta}_{\underline{\alpha}}\right)^2, \tag{29}$$

where

$$\tilde{c} = \tilde{W}_t\cdot\hat{n} + \varepsilon^3\tilde{\delta}(\tilde{W}_{\underline{\alpha}}\cdot\hat{n}) + \varepsilon^3\tfrac{\tilde{\gamma}}{2}\tilde{\theta}_t + \varepsilon^6\tfrac{\tilde{\gamma}}{2}\tilde{\delta}\tilde{\theta}_{\underline{\alpha}} + (cos(\varepsilon^3\tilde{\theta}) - 1)\,, \tag{30}$$

$$\tfrac{1}{2}\tilde{\gamma} + \tilde{W}\cdot\hat{t} = \tilde{v}\,, \tag{31}$$

$$\tilde{\delta} = \tilde{v} - \varepsilon^3\tilde{T}\,. \tag{32}$$

Theorem 2.1: (Theorem 1.1 in the arc length formulation)

For all $b_0, C_0, \tau_0 > 0$ there exist an $\varepsilon_0 > 0$ such that for all $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leq \varepsilon_0$ and all $b \in \mathbb{R} \setminus \{\frac{1}{3}\}$ with $0 \leq b \leq b_0$ the following is true. Let

$$\tilde{y}|_{t=0}(\underline{\alpha}) = \tilde{\Phi}_1(\underline{\alpha}), \qquad \tilde{v}|_{t=0}(\underline{\alpha}) = \tilde{\Phi}_2(\underline{\alpha})$$

with $\|(\tilde{\Phi}_1, \tilde{\Phi}_2)\|_{H^{s+8}_{\underline{\alpha}} \cap H^{s+3}_{\underline{\alpha}}(k)} \leq C_0 \varepsilon^l$, where $s \geq 7$, $k > 1$ and $l \geq 0$. Let

$$(A_1)_\tau = \left(\tfrac{b}{2} - \tfrac{1}{6}\right)(A_1)_{\underline{\alpha}\underline{\alpha}\underline{\alpha}} - \tfrac{3}{2} A_1 (A_1)_{\underline{\alpha}}, \qquad (A_2)_\tau = \left(\tfrac{1}{6} - \tfrac{b}{2}\right)(A_2)_{\underline{\alpha}\underline{\alpha}\underline{\alpha}} + \tfrac{3}{2} A_2 (A_2)_{\underline{\alpha}},$$

$$A_1|_{\tau=0} = \tfrac{1}{2}(\tilde{\Phi}_1 + \tilde{\Phi}_2), \qquad A_2|_{\tau=0} = \tfrac{1}{2}(\tilde{\Phi}_1 - \tilde{\Phi}_2).$$

Then there exists a unique solution of the 2–D water wave problem (26)–(29) with the above initial conditions satisfying

$$\sup_{t \in [0, \tau_0/\varepsilon^3]} \left\| \begin{pmatrix} \tilde{y} \\ \tilde{v} \end{pmatrix} (\cdot, t) - \psi(\cdot, t) \right\|_{H^s_{\underline{\alpha}} \times H^{s-1/2}_{\underline{\alpha}}} \lesssim \varepsilon^{2+l}$$

where

$$\psi(\underline{\alpha}, t) = A_1\left(\underline{\alpha} - \varepsilon t, \varepsilon^3 t\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \varepsilon^2 A_2\left(\underline{\alpha} + \varepsilon t, \varepsilon^3 t\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

3. Main Steps of the Proof

- *Step 1:* Find explicit expressions for the linear and the quadratic terms of the system (26)–(29) and bounds for the cubic and higher order terms.
  Main tools: 1. Taylor expansions of the Birkhoff–Rott Integral and its derivatives. 2. Find the right balance between size and regularity.
- *Step 2:* Refind the KdV–equation approximately in (26)–(29).
- *Step 3:* Write the exact solutions of (26)–(29) as approximation plus error and construct a suitable nonlinear energy being equivalent to the square of a Sobolev–norm to estimate the error on a timespan of order $\mathscr{O}(\varepsilon^{-3})$ w.r.t. t.
- *Step 4:* Express the proven result in Eulerian coordinates.
- Treat the Kawahara case analogously.

- To Step 1: We obtain

$$
\begin{aligned}
\tilde{y}_t &= K_0(\tilde{v}) + \varepsilon^3\big(K_0[K_0,\tilde{y}]\tilde{v} - (1+K_0^2)(\tilde{y}\tilde{v})\big)_{\underline{\alpha}} + h.o.t.,\\
\tilde{v}_t &= -\varepsilon\tilde{y}_{\underline{\alpha}} + \varepsilon^3 b\,\tilde{y}_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}} - \varepsilon^3\tilde{\delta}\tilde{v}_{\underline{\alpha}} + \varepsilon^3 K_0(\tilde{\delta}_{\underline{\alpha}})K_0(\tilde{v}) + h.o.t.,\\
\tilde{\theta}_t &= K_0(\tilde{\delta}_{\underline{\alpha}}) - \varepsilon^3\tilde{\delta}\,\tilde{\theta}_{\underline{\alpha}} + \varepsilon^3\big(K_0[K_0,\tilde{y}]\tilde{\delta}_{\underline{\alpha}} - (1+K_0^2)(\tilde{y}\tilde{\delta}_{\underline{\alpha}})\big)_{\underline{\alpha}}\\
&\quad + \varepsilon^3\big(K_0[K_0,\tilde{\theta}]\tilde{\delta}_{\underline{\alpha}} - (1+K_0^2)(\tilde{\theta}\,\tilde{\delta}_{\underline{\alpha}})\big) + h.o.t.,\\
\tilde{\delta}_{\underline{\alpha}t} &= -\varepsilon\big(1 - \varepsilon^3 K_0(\tilde{\theta}) + \varepsilon^5 b\,K_0(\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}}) + h.o.t.\big)\tilde{\theta}_{\underline{\alpha}} + \varepsilon^3 b\,\tilde{\theta}_{\underline{\alpha}\,\underline{\alpha}\,\underline{\alpha}}\\
&\quad - \varepsilon^3(\tilde{\delta}\,\tilde{\delta}_{\underline{\alpha}})_{\underline{\alpha}} + \varepsilon^3(K_0(\tilde{\delta}_{\underline{\alpha}}))^2 + h.o.t.,\\
\tilde{\delta}(\underline{\alpha},t) &= \tilde{v}(\underline{\alpha},t) - \varepsilon^3\int_{-\infty}^{\underline{\alpha}}(K_0(\tilde{v})\,\tilde{\theta}_{\underline{\alpha}})(\underline{\beta},t)\,d\underline{\beta} + h.o.t.,
\end{aligned}
$$

where

$$\hat{K}_0(\underline{k}) = -i\tanh(\varepsilon\underline{k})\,.$$

- To Step 3: Let

$$\begin{aligned}
\tilde{y}(\underline{\alpha},t) &= A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)+A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{y}}(\underline{\alpha},t)\,,\\
\tilde{v}(\underline{\alpha},t) &= A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)-A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{v}}(\underline{\alpha},t)\,,\\
\tilde{\theta}(\underline{\alpha},t) &= \partial_{\underline{\alpha}}A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)+\partial_{\underline{\alpha}}A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{\theta}}(\underline{\alpha},t)\,,\\
\tilde{\delta}_{\underline{\alpha}}(\underline{\alpha},t) &= \partial_{\underline{\alpha}}A_1(\underline{\alpha}-\varepsilon t,\varepsilon^3 t)-\partial_{\underline{\alpha}}A_2(\underline{\alpha}+\varepsilon t,\varepsilon^3 t)+\varepsilon^2 R_{\tilde{\delta}_{\underline{\alpha}}}(\underline{\alpha},t)\,.
\end{aligned}$$

Then the error $R=(R_{\tilde{y}},R_{\tilde{v}},R_{\tilde{\theta}},R_{\tilde{\delta}_{\underline{\alpha}}})$ satisfies

$$\begin{aligned}
\partial_t R_{\tilde{y}} &= K_0 R_{\tilde{v}}+\varepsilon^3\mathscr{N}_1\,,\\
\partial_t R_{\tilde{v}} &= -\varepsilon\,\partial_{\underline{\alpha}}R_{\tilde{y}}+\varepsilon^3 b\,\partial_{\underline{\alpha}}^3 R_{\tilde{y}}+\varepsilon^3\mathscr{N}_2\,,\\
\partial_t R_{\tilde{\theta}} &= K_0 R_{\tilde{\delta}_{\underline{\alpha}}}-\varepsilon^3\tilde{\delta}\,\partial_{\underline{\alpha}}R_{\tilde{\theta}}-\varepsilon^3\partial_{\underline{\alpha}}(1+K_0^2)(\tilde{y}R_{\tilde{\delta}_{\underline{\alpha}}})+\varepsilon^3\mathscr{N}_3\,,\\
\partial_t R_{\tilde{\delta}_{\underline{\alpha}}} &= -\varepsilon(1+\varepsilon^3 C_R)\,\partial_{\underline{\alpha}}R_{\theta}+\varepsilon^3 b\,\partial_{\underline{\alpha}}^3 R_{\tilde{\theta}}-\varepsilon^6 b\,(\partial_{\underline{\alpha}}\tilde{\theta})K_0\partial_{\underline{\alpha}}^2 R_{\tilde{\theta}}\\
&\quad -\varepsilon^3\tilde{\delta}\,\partial_{\underline{\alpha}}R_{\tilde{\delta}_{\underline{\alpha}}}+\varepsilon^3\mathscr{N}_4\,.
\end{aligned}$$

- We use the following energy:

$$\mathscr{E}(t) = E(t) + E_b(t) + \sum_{k=0}^{s} E_k(t) + \sum_{k=0}^{s} E_{b,k}(t)$$

for $s \geq 6$, where

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} R_{\tilde{y}} K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}}) R_{\tilde{y}}\, d\underline{\alpha} + \frac{1}{2}\int_{\mathbb{R}} R_{\tilde{v}}^2\, d\underline{\alpha}\,,$$

$$E_b(t) = \frac{b}{2}\varepsilon^2 \int_{\mathbb{R}} (\partial_{\underline{\alpha}} R_{\tilde{y}}) K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}})(\partial_{\underline{\alpha}} R_{\tilde{y}})\, d\underline{\alpha}\,,$$

$$E_k(t) = \frac{1}{2}\int_{\mathbb{R}} (1+\varepsilon^3 C_R)(\partial_{\underline{\alpha}}^k R_{\tilde{\theta}}) K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}})(\partial_{\underline{\alpha}}^k R_{\tilde{\theta}})\, d\underline{\alpha} + \frac{1}{2}\int_{\mathbb{R}} (\partial_{\underline{\alpha}}^k R_{\tilde{\delta}_{\underline{\alpha}}})^2\, d\underline{\alpha}$$

$$+\frac{1}{2}\varepsilon^2 \int_{\mathbb{R}} (\partial_{\underline{\alpha}}^k R_{\tilde{\delta}_{\underline{\alpha}}}) K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}})(1+K_0^2)(\tilde{y}\,\partial_{\underline{\alpha}}^k R_{\tilde{\delta}_{\underline{\alpha}}})\, d\underline{\alpha}\,,$$

$$E_{b,k}(t) = \frac{b}{2}\varepsilon^2 \int_{\mathbb{R}} (\partial_{\underline{\alpha}}^{k+1} R_{\tilde{\theta}}) \big(K_0^{-1}(-\varepsilon\partial_{\underline{\alpha}}) + \varepsilon^4(\partial_{\underline{\alpha}}\tilde{\theta}) + \varepsilon^6(\partial_{\underline{\alpha}} R_{\tilde{\theta}})\big)(\partial_{\underline{\alpha}}^{k+1} R_{\tilde{\theta}})\, d\underline{\alpha}\,.$$

- We show:

$$\frac{d}{dt}\mathscr{E} \lesssim \varepsilon^3(\mathscr{E}+1)$$

  uniformly w.r.t. all $b \in \mathbb{R}_0^+ \setminus \{\frac{1}{3}\}$ with $b \leq b_0$.

  Main ingredients of the argumentation:

  – The energy $\mathscr{E}$ is constructed in a such way that all terms in the error equations that cannot be estimated directly cancel.

  – Transport terms do not cause a loss of regularity.

  – Use of commutator estimates.

- Now, an application of Gronwall's inequality yields the boundedness of the error on the right time scale.

For further details:

W.-P. Düll. Validity of the Korteweg-de Vries Approximation for the Two-Dimensional Water Wave Problem in the Arc Length Formulation. *Comm. Pure Appl. Math.* **65** (2012), no. 3, 381-429.