Existence of N-solitary waves for the fractional Korteweg-de Vries equation

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Outline

1. The fractional Korteweg-de Vries equation
   - Well-posedness
   - Properties of the solitary waves

2. Weakly interacting solitary waves
   - N-solitary waves for fKdV
   - Martel-Merle's method
   - Control of the geometric parameters
   - Commutator estimates

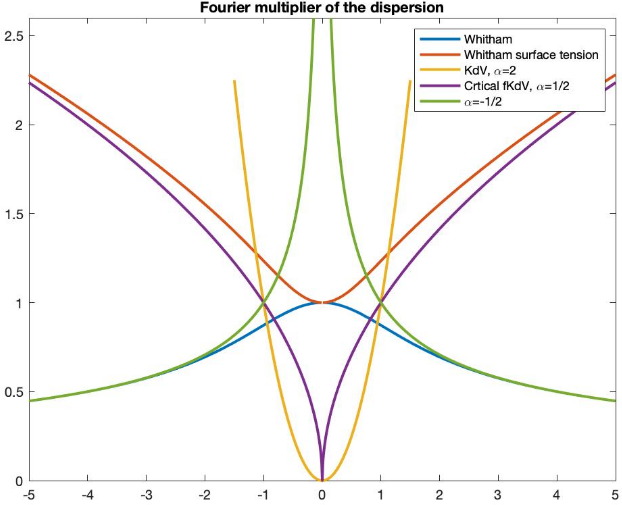
The fractional Korteweg-de Vries equation

The fractional Korteweg-de Vries equation ($\alpha \leq 2$):

$$\partial_t u(t,x) - \partial_x |D|^\alpha u(t,x) + \partial_x(u^2)(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}$$

with $\mathcal{F}\left(|D|^\alpha u\right)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi)$.
Physical equations:

- $\alpha = 2$, Korteweg-de Vries equation,
- $\alpha = 1$, Benjamin-Ono equation,
- $\alpha = \frac{1}{2}, -\frac{1}{2}$ apprixImation in high frequency of the Whitham equation with and without surface tension.


Fourier multiplier of the dispersion
Whitham
Whitham surface tension
KdV, α=2
Crtical fKdV, α=1/2
α=-1/2
2.5
2
1.5
1
0.5
0
-5
-4
-3
-2
-1
0
1
2
3
4
5

The fractional Korteweg-de Vries equation

Except for $\alpha = 2$ (KdV), $\alpha = 1$ (BO), the equation is not known to be completely integrable. We still have two conserved quantities:

$$M(u) = \int u^2, \quad E(u) = \int \frac{u|D|^\alpha u}{2} - \frac{u^3}{3}.$$

If u is a solution then $\forall c > 0, u_c(t,x) = cu\left(c^{\frac{1+\alpha}{\alpha}} t, c^{\frac{1}{\alpha}} x\right)$.
The equation is L^2-sub-critical if $\alpha > \frac{1}{2}$.
The equation is globally well-posed:

- in $L^2(\mathbb{R})$ for $1 < \alpha < 2$ (Herr-Ionescu-Kenig-Koch 2009),
- in $L^2(\mathbb{R})$ for $\alpha = 1$ (Ionescu-Kenig 2007, Tao 2004),
- in $H^{\frac{\alpha}{2}}(\mathbb{R})$ for $\frac{6}{7} < \alpha < 1$ (Molinet-Pilod-Vento 2018),
- and it is conjectured in $H^{\frac{\alpha}{2}}(\mathbb{R})$ for $\frac{1}{2} < \alpha \leq \frac{6}{7}$ (Klein-Saut 2015).

Definitions

Definition of the important objects:

- A **solitary wave** moving at speed $c > 0$ is defined as a solution of the KdV equation on the form :

$$Q_c(x - ct),$$

  with $Q_c(x) \to 0, \quad x \to \pm\infty.$
- A N-**solitary wave** is a solution behaves at infinity like a sum of N decoupled solitary waves:
  - weakly interacting: all different speeds.
  - strongly interacting: relative distance $\sim t^\beta$, with $\beta < 1$.

The solitary waves and N-solitary waves are universal objects:

- fluid mechanics,
- quantum mechanics,
- biology.

Properties of solitary waves

Theorem

Let $\alpha \in \left(\frac{1}{3}, 2\right)$. There exists $Q \in H^{\frac{\alpha}{2}}(\mathbb{R}) \cap C^{\infty}(\mathbb{R})$ such that

1. *(Existence) The function Q solves $|D|^{\alpha} Q + Q - Q^2 = 0$ and $Q = Q(|x|) > 0$ is even, positive and strictly decreasing in $|x|$.*
2. *(Uniqueness) The even ground state solution $Q = Q(|x|) > 0$ is unique.*
3. *(Decay) The function Q verifies the following decay estimate*

$$Q^{(k)}(x) \sim_{\pm\infty} x^{-\alpha-1-k}.$$

4. *(Stability) The function Q is orbitally stable in $H^{\frac{\alpha}{2}}(\mathbb{R})$.*

Properties of solitary waves

1. *Particular case* $\alpha = 2, \alpha = 1$ corresponds respectively to the KdV equation and to the Benjamin-Ono equation

$$Q_{KDV}(x) = \frac{3}{2}\cosh^{-2}\left(\frac{x}{2}\right), \quad Q_{BO}(x) = \frac{4}{1+x^2}.$$

Uniqueness of the solitary wave of the Benjamin-Ono equation obtained by Benjamin 1967, Amick-Toland 1991, Albert 1995. Uniqueness based on harmonic extension process:

- P poisson kernel.
- $U(x,y) = P(\cdot, y) *_x u$ is harmonic, and $\lim_{y\to 0} \partial_y U(x,y) = |D|^1 u$.
- $|D|^1 Q + Q - Q^2 = 0 \longrightarrow \Delta U = 0, \partial_y U(x,0) = Q^2 - Q$.

General case $\alpha \in \left(\frac{1}{3}, 2\right)$.

- Existence (positive) was proved by Weinstein 1987, Albert-Bona-Saut 1997:

$$\inf_{u \in H^{\frac{\alpha}{2}}} \frac{\left(\int ||D|^{\frac{\alpha}{2}} u|^2\right)^{\frac{1}{2\alpha}} \left(\int |u|^2\right)^{\frac{\alpha-1}{2\alpha+1}}}{\int |u|^3} \Rightarrow |D|^\alpha Q + Q = Q^2.$$

- Even: Moving plane method (Alexandrov 1962, Serrin 1971, Gidas-Ni-Nirenberg 1979, 1981).
- Smoothness: Consequence of Kato-Ponce estimate (Grafakos-Oh 2013), or compact injections.

2. *Uniqueness:*
   - The solitary waves of KdV/Schrodinger is unique (Kwong 1989)

$$\Delta Q - Q + Q^p = 0.$$

   - For $\alpha \neq 2$, the ground state (solution of the minimising problem) is unique (Frank-Lenzmann 2013), based on an extension process introduced by Caffarelli-Silvestre 2007 and on the understanding spectrum of $\mathcal{L} = |D|^\alpha + 1 - 2Q^2$.

Properties of solitary waves

3. First asymptotic order of the groud state given by Frank-Lenzmann-Silvestre 2016, Kenig-Martel-Robbiano 2011.

Theorem (E-Valet, 2023 JDE)

Let $\alpha \in \left(\frac{1}{3}, 2\right]$*:*

$$Q(x) - \frac{a_1}{x^{\alpha+1}} - \frac{a_2}{x^{2\alpha+1}} = o_{+\infty}\left(\frac{1}{x^{2\alpha+1}}\right)$$

$$Q^{(j)}(x) - (-1)^j \frac{(\alpha+j)!}{\alpha!} \frac{a_1}{x^{\alpha+1+j}} = o_{+\infty}\left(\frac{1}{x^{\alpha+1+j}}\right).$$

Higher dimension: Frank-Lenzmann-Silvestre 2016, Riano-Roudenko 2023

Proof of the asymptotic

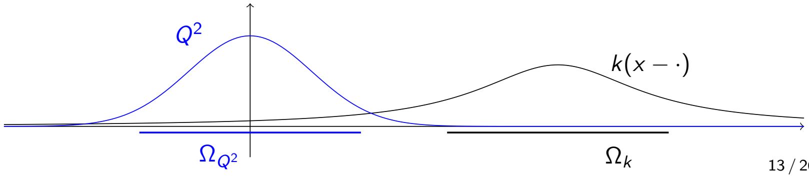
From the equation of Q, we have:

$$|D|^\alpha Q + Q = Q^2 \Rightarrow Q(x) = (|D|^\alpha + 1)^{-1} Q^2(x).$$

We define $k(x) = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^\alpha}\right)$. Therefore:

$$Q(x) = k \star Q^2(x) = \int_{\Omega_k} k(x-y)Q^2(y)dy + \int_{\Omega_k^c} k(x-y)Q^2(y)dy.$$

- If Q vanishing then Q should have at least an algebraic decay (Bona-Li 1997).
- Decay of k related to the regularity of $|D|^\alpha$, we study the function k thanks to **Pólya 1923** (complex integration).



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N-solitary waves

Construction of non explicit N-solitary waves:

- the non-linear damped wave equations: Feireisl 1998.
- the gKdV equation: Martel 2005 ($p \leq 5$) based on Merle 1993, Côte-Martel-Merle 2011 ($p > 5$), Nguyen 2019.
- the NLS family equation: Martel-Merle 2006, Côte-Martel-Merle 2011, Martel-Nguyen 2019, Nguyen 2020, Ferriere 2021. Based on a fixed point method: Le Coz-Li-Tsai 2015, Van Tin 2021.
- the Klein-Gordon equation: Côte-Muñoz 2014, Bellazzini-Ghimenti-Le Coz 2014, Côte, Martel 2018, Aryan 2021.
- the wave equation: Martel-Merle 2016, Jendrej-Martel 2020 .
- the water-waves equation: Ming-Rousset-Tzvetkov 2015.
- the Zakharov-Kuznetsov equation: Valet 2021.

N-solitary waves for fKdV

Theorem (E. *Revista Matemática Iberoamericana* 2022)

We assume $\alpha \in \left(\frac{1}{2}, 2\right)$. *Let* $N \in \mathbb{N}$, $0 < c_1 < \cdots < c_N < +\infty$. *Then, there exist some constants* $T_0 > 0$, $C_0 > 0$, N *functions* $\rho_1, \cdots, \rho_N \in C^1([T_0, +\infty))$ *and* $U \in C^0([T_0, +\infty) : H^{\frac{\alpha}{2}}(\mathbb{R}))$ *solution of (fKdV) such that, for all* $t \geq T_0$,

$$\left\| U(t,\cdot) - \sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_j(t)) \right\|_{H^{\frac{\alpha}{2}}} \leq \frac{C_0}{t^{\frac{\alpha}{2}}},$$

$$|\rho_j(t) - c_j t| \leq t^{1-\frac{\alpha}{4}} \quad \text{and} \quad |\rho_j'(t) - c_j| \leq \frac{C_0}{t^{\frac{\alpha}{2}}},$$

for all $j \in \{1, \cdots, N\}$.

Martel-Merle's method

$$S_n \nearrow +\infty \quad , \quad \rho_{j,n} \sim c_j S_n.$$

$$u_n(T_0,x) \sim \sum_{j=1}^{N} Q_{c_j}(x - c_j T_0) \qquad u_n(S_n,x) = \sum_{j=1}^{N} Q_{c_j}(x - \rho_{j,n}).$$

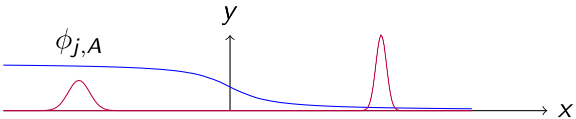
Martel-Merle's method

$$\left\| u_n(t,\cdot) - \sum_{j=1}^{N} Q_{c_j}(\cdot - \rho_{j,n}(t)) \right\|_{H^{\frac{\alpha}{2}}} \leq \frac{C}{t^{\frac{\alpha}{2}}}, \quad \forall t \in [T_0, S_n]$$

1. Coercivity of the linearized operator Frank-Lenzmann 2013.
2. Monotonicity of the localized mass and energy

$$M_j = \int u^2 \phi_{j,A}, \quad E_j = \int \left(\frac{u|D|^{\alpha} u}{2} - \frac{u^3}{3} \right) \phi_{j,A}.$$

With $\phi_{j,A}(x) = \phi\left(\frac{x - \frac{\rho_{j,n}+\rho_{j+1,n}}{2}}{A} \right)$:



Difficulties

- Estimate for the localized mass Kenig-Martel-Robbiano. Extend commutator estimate introduced by Kenig-Martel.
- We extend in a non symmetric case the estimate developed by Kenig-Martel-Robbiano.
- $[|D|^\alpha, \phi_A] = [|D|^\alpha \chi(D), \phi_A] + [|D|^\alpha (1 - \chi(D)), \phi_A]$.
- Non smoothness of the operator $|D|^\alpha \Rightarrow$ strong restriction on the decay of ϕ.
- Choose carefully the initial data, to integrate the quantity

$$|\rho'_{j,n}(t) - c_j| \leq \frac{C}{t^{\frac{\alpha}{2}}}, \quad \forall t \in [T_0, S_n].$$

Control of the geometric parameters

- The functions ρ_j are introduced to get the coercivity of the linearized operator.
- The parameters ρ_j' are controlled by $u - \sum Q_{c_j}$:

$$|\rho_{j,n}'(t) - c_j| \leq \frac{C}{t^{\frac{\alpha}{2}}} \quad \not\Rightarrow \quad |\rho_{j,n}(t) - c_j t| \leq C t^{1-\frac{\alpha}{2}}.$$

- We replace the usual initial condition $\rho_{j,n}(S_n) = \rho_{j,n} = c_j S_n$ by $\rho_{j,n} = c_j S_n + \lambda_{j,n} S_n^{1-\frac{\alpha}{4}}$, with $\lambda_{j,n} \in [-1,1]$ and the bootstrap by $|\rho_{j,n}(t) - c_j t| \leq t^{1-\frac{\alpha}{4}}$.
- Study of:

$$\Phi : [-1,1]^N \to \partial[-1,1]^N$$
$$\lambda \mapsto ((\rho_{j,n}(t^*(\lambda)) - c_j t^*(\lambda))(t^*)^{\frac{\alpha}{4}-1}(\lambda))_{j=1}^N.$$

Commutator estimates

$$\frac{1}{2}\frac{d}{dt}\int u^2\phi_{j,A} = \int |D|^\alpha u\left(-\partial_x u\phi_{j,A} + u|\phi'_{j,A}|\right) - \frac{u^3}{3}|\phi'_{j,A}| + \frac{\rho'_j + \rho'_{j+1}}{2}\int u^2|\phi'_j|.$$

Let $\alpha \in (0,2)$. In the symmetric case, there exists $C > 0$ such that

$$\left|\int \left(|D|^\alpha u\right) u|\phi'_{j,A}| - \int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\phi'_{j,A}|}\right)\right)^2\right| \leq \frac{C}{A^\alpha}\int u^2|\phi'_{j,A}|,$$

and

$$\left|\int \left(|D|^\alpha u\right)\partial_x u\phi_{j,A} + \frac{\alpha-1}{2}\int \left(|D|^{\frac{\alpha}{2}}\left(u\sqrt{|\phi'_{j,A}|}\right)\right)^2\right| \leq \frac{C}{A^\alpha}\int u^2|\phi'_{j,A}|,$$

for any $u \in \mathcal{S}(\mathbb{R})$, $A > 1$ and $j \in \{1, \cdots, N\}$.

Pseudo-differential toolbox

We define the symbolic class $S^{m,q}(\mathbb{R}^2)$ by

$$S^{m,q}(\mathbb{R}^2) = \left\{ \begin{array}{l} a \in C^\infty(\mathbb{R}^2), \forall k, \beta \in \mathbb{N}, \exists C_{k,\beta} > 0 : \\ |\partial_x^k \partial_\xi^\beta a(x,\xi)| \leq C_{k,\beta} \langle x \rangle^{q-k} \langle \xi \rangle^{m-\beta} \end{array} \right\}$$

We define the operator associated to a by the following formula for $u \in \mathcal{S}(\mathbb{R})$

$$a(x,D)u = \frac{1}{2\pi} \int e^{ix\xi} a(x,\xi) \mathcal{F}(u)(\xi) d\xi.$$

The pseudo-differential operator enjoy some algebraic properties!

Pseudo-differential toolbox

Let $a(x,D), b(x,D)$ two pseudo-differential operators associated to the symbol $a \in \mathcal{S}^{m,q}$ and $b(x,\xi) \in \mathcal{S}^{m',q'}$. Then

1. **Weighted operator:** there exists $C > 0$, such that for all $u \in \mathcal{S}$, $\|a(x,D)u\|_{L^2} \leq C\|\langle x\rangle^q \langle D\rangle^m u\|_{L^2}$, with $\langle D\rangle = (1+|D|^2)^{\frac{1}{2}}$.
2. **Composition:** $a(x,D) \circ b(x,D)$ is a pseudo-differential operator and there exists $c(x,\xi) \in \mathcal{S}^{m+m',q+q'}$ such that c is the symbol associated to $a(x,D) \circ b(x,D)$.
3. **Commutator:** there exist an operator $c(x,D)$ such that the symbol $c \in \mathcal{S}^{m+m'-1,q+q'-1}$ and $[a(x,D), b(x,D)] = c(x,D)$.
4. **Taylor expansion:** If a^* is adjoint of a then $$a^*(x,\xi) = \sum_{\beta \leq k} \frac{1}{\beta!} \partial_\xi^\beta \partial_x^\beta \bar{a}(x,\xi) + R_k(x,\xi),$$ with $\partial_\xi^\beta \partial_x^\beta \bar{a} \in \mathcal{S}^{m-\beta,q-\beta}$ and $R_k \in \mathcal{S}^{m-k-1,q-k-1}$ explicit.

Pseudo-differential toolbox

High frequency estimates:

$$\|[(1-\chi(D))|D|^{\alpha},\phi]u\|_{L^2} \leq C\|\sqrt{|\phi|'}\langle D\rangle^{\alpha-1}u\|_{L^2}.$$

$(1-\chi(D))|D|^{\alpha}$ symbol belongs to $\mathcal{S}^{\alpha,0}$, $\phi(x)$ symbol belongs to $\mathcal{S}^{0,0}$. This implies the symbol of $[(1-\chi(D))|D|^{\alpha},\phi]$ belongs to $\mathcal{S}^{\alpha-1,-1}$. From the Taylor expansion we obtain $\mathcal{S}^{\alpha-1,-\beta}$, with β such that $\phi' \sim_{\pm\infty} x^{-\beta}$.

No restriction from the high frequency.

Schur's lemma

Low frequency estimates: Let $k(x) = \mathcal{F}^{-1}(\chi(\xi)|\xi|^\alpha)$.

$$[\chi(D)|D|^\alpha, \phi]u = \int_{\mathbb{R}} k(x-y)\,(\phi(x)-\phi(y))\,u(y)dy$$
$$= \int_{\mathbb{R}} \underbrace{k(x-y)\,(\phi(x)-\phi(y))\,\sqrt{|\phi'|^{-1}(y)}}_{=:K(x,y)} \left(\sqrt{|\phi'|(y)}u(y)\right) dy$$

Question: for which ϕ, K is a kernel of a bounded operator in $L^2(\mathbb{R})$.

Answer: $\phi(x) = \int_x^\infty \langle y\rangle^{-r} dy$, with $r \in \left(\frac{1}{2}, 1+\alpha\right]$.

Applying the Schur's test :

$$\|[\chi(D)|D|^\alpha, \phi]u\|_{L^2} \leq C\|u\sqrt{|\phi|'}\|_{L^2}.$$

Thank you!