Guaranteed lift-off in non-Newtonian thin-film equations

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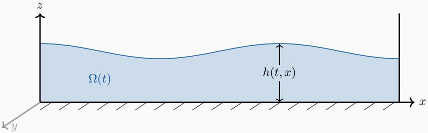
Non-Newtonian thin films

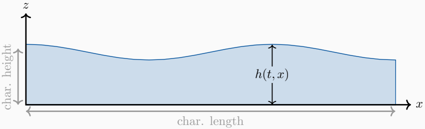
Thin fluid film

- incompressible
- viscous

$$\mu(\epsilon) = \mu_0 |\epsilon|^{\frac{1}{\alpha}-1}$$

- capillary-driven





Navier-Stokes system: $\vec{u} = (u, v)$ velocity field in $\Omega(t)$

$$\begin{cases} \rho(\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u}), & = & \mathrm{div} \cdot T(p, \vec{u}), & (t,x) \in \Omega(t) \\ \mathrm{div}\, \vec{u} & = & 0, & (t,x) \in \Omega(t) \\ T(p, \vec{u})\vec{n} & = & \gamma\kappa\vec{n}, & z = h(t,x) \\ \partial_t h + u\partial_x h & = & v & z = h(t,x) \\ \vec{u} & = & 0, & z = 0 \end{cases}$$

Properties

Thin-film equation

$$\begin{cases} \partial_t h + \partial_x (h^n |\partial_x^3 h|^{\alpha-1} \partial_x^3 h) &= 0, & t > 0, x \in (0,1) \\ \partial_x h = h^n |\partial_x^3 h|^{\alpha-1} \partial_x^3 h &= 0, & t > 0, x \in \partial(0,1) \\ h(0,x) &= h_0(x), & x \in (0,1) \end{cases}$$

- fourth-order doubly degenerate-parabolic equation
- mass conservation: $\int_{(0,1)} h(t,x)\,\mathrm{d}x = \int_{(0,1)} h_0(x)$ă $\mathrm{d}x$
- driven by surface tension: forces equilibrate surface area

$$E_s[h] = \int_{(0,1)} \sqrt{1 + |\partial_x h|^2}\,\mathrm{d}x \cong 1 + \frac{1}{2} \int_{(0,1)} |\partial_x h|^2\,\mathrm{d}x = 1 + E[h]$$

- energy-dissipation mechanism

$$\frac{d}{dt} E[h](t) = -\int_{(0,1)} h^n |\partial_x^3 h|^{\alpha+1}\,\mathrm{d}x \le 0$$

Weak solutions

Thin-film equation

$$\begin{cases} \partial_t h + \partial_x(h^n|\partial_x^3 h|^{\alpha-1}\partial_x^3 h) &= 0, \quad t>0, x\in(0,1) \\ \partial_x h = h^n|\partial_x^3 h|^{\alpha-1}\partial_x^3 h &= 0, \quad t>0, x\in\partial(0,1) \\ h(0,x) &= h_0(x), \quad x\in(0,1) \end{cases}$$

Definition

$h \in L_\infty([0,\infty); H^1(\Omega)) \cap C^{\frac{1}{5\alpha+3},\frac{1}{2}}([0,\infty]\times\bar{\Omega})$ with $\partial_x^3 h \in L_{\alpha+1,\mathrm{loc}}(\{u>0\})$ and $\partial_t h \in L_{\alpha+1}([0,\infty); (W^1_{\alpha+1}(\Omega))')$ is a weak solution to the thin-film equation if

$$\int_0^\infty \langle \partial_t h, \phi\rangle_{W^1_{\alpha+1}}\,\mathrm{d}t - \iint_{\{u>0\}} h^n|\partial_x^3 h|^{\alpha-1}\partial_x^3 h \partial_x\phi\,\mathrm{d}x\,\mathrm{d}t = 0$$

for all $\phi \in L_{\alpha+1}((0,\infty); W^1_{\alpha+1,B}(\Omega))$.

Non-negativity of solutions

Question

Do solutions remain non-negative?

- For $\partial_t u + \Delta^2 u = 0$, the answer is no!
- If $\alpha = 1$ and $n > 1$, use entropy method:

$$\int_{(0,1)} g(h(t,x))\,\mathrm{d}x + \int_0^t \int_{(0,1)} |\partial_x^2 h|\,\mathrm{d}x\,\mathrm{d}t \le \int_{(0,1)} g(h(0,x))\,\mathrm{d}x,$$

where $g''(h) = \frac{1}{h^n}$, hence $g(h) \sim h^{2-n}$ and $g(h) = +\infty,\ h < 0$

- For non-Newtonian rheologies: regularisation techniques

Existence of weak solutions

Theorem

Let $h_0 \in H^1((0,1))$, $h_0 \geq 0$. Then there exists a weak solution h to the thin-film equation that satisfies

$$E[h](t) + \int_0^t \int_{(0,1)} h^n |\partial_x^3 h|^{\alpha+1} \, \mathrm{d}x \, \mathrm{d}s \leq E[h_0] \tag{EDI}$$

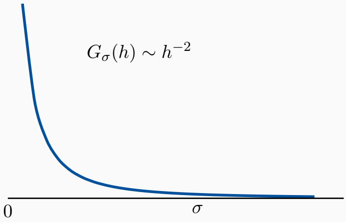
If $h_0 > 0$, then (EDI) is an equality as long as $h(t,\cdot) > 0$.

- Positivity by singular potential:

$$\partial_x^3 h \to \partial_x^3 h - G_\sigma''(h)\partial_x h$$

$$E[h] \to E[h] + \int G_\sigma(h) \, \mathrm{d}x$$

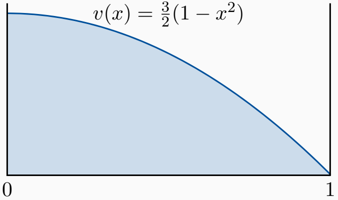
- Gradient-flow approach in $H^1(\Omega)$
- energy methods to identify limit as $\sigma \to 0$



The lift-off property

- $v(x) = \frac{3}{2}(1 - x^2)$ is steady state

$$\int_{(0,1)} v(x)\,\mathrm{d}x = 1$$



- Observation

$$v = \arg\min E[h]$$

  among all $h \geq 0$ with $\int h\,\mathrm{d}x = \int v\,\mathrm{d}x = 1$ and at least one root
- let $E[h_0] < E[v]$ and $\int h_0\,\mathrm{d}x = 1 \Rightarrow E[h](t) < E[v]$ for all t
- Conclusion: h solution with initial value h_0, then $h(t) > 0$ and

$$E[h](t) + \int_s^t \int_{(0,1)} h^n |\partial_x^3 h|^{\alpha+1}\,\mathrm{d}x\,\mathrm{d}\tau = E[h](s)$$

The lift-off property

Question

Is the dissipation $\int_{(0,1)} h^n |\partial_x^3 h|^{\alpha+1} \, \mathrm{d}x$ of order one?

Theorem

Let $n, \alpha > 0$ with $2(\alpha + 1) > n$. Then there exists $t_0 = t_0(n, \alpha) > 0$ such that whenever $h_0 \in H^1(\Omega)$ with $\int_{(0,1)} h_0 \, \mathrm{d}x = 1$ and $E[h_0] < E[v]$, then any weak solution h to the thin-film equation with initial value h_0 satisfies

$$\min_{x \in \bar{\Omega}} h(t,x) \geq \frac{1}{2}, \quad t \geq t_0.$$

Long-time behaviour: convergence to mean with explicit convergence rates

Idea of proof

Lemma

Let $n, \alpha > 0$. There is a constant $C = C(n, \alpha) > 0$ such that for any $h \in W^3_{\alpha+1}(\Omega)$ with $h'(0) = h'(1) = 0$, $\int h \,\mathrm{d}x = 1$ and $\min_{x \in \bar\Omega} h = \delta \in (0, 1/2)$ we have

$$\int_\Omega h^n |\partial_x^3 h|^{\alpha+1} \,\mathrm{d}x \geq \frac{C}{\log^{\alpha+1}(M/\delta)} \min\{\delta^{n-1-2\alpha}, 1\}.$$

- $\min h = \delta < \frac{1}{2} \Rightarrow \max h > 1 \Rightarrow \exists x_0 : |h'(x_0)| > \frac{1}{4}$ and $h''(x_0) = 0$
- from $h'(1) = 0$, we conclude there is x_1 with $|h''(x_1)| = 1/2$
- By Hölder's inequality

$$\begin{aligned}(h''(x_0) + \frac{1}{2})^{\alpha+1} &= \left(\int_{x_0}^{x_1} -\partial_x^3 h \,\mathrm{d}x\right)^{\alpha+1} \\ &\leq \left(\int_{x_0}^{x_1} h^n |\partial_x^3 h|^{\alpha+1} \,\mathrm{d}x\right) \left(\int_{x_0}^{x_1} h^{-\frac{n}{\alpha}} \,\mathrm{d}x\right)^{\alpha}\end{aligned}$$

Idea of proof

Statement of Theorem

Want: $\min_{x\in\bar{\Omega}} h(t,x) \geq \frac{1}{2}, \quad t \geq t_0$

- As long as $\min_{x\in\bar{\Omega}} h(t,x) \leq \frac{1}{2}$:

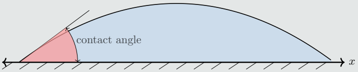
$$\begin{aligned} \min_{x\in\bar{\Omega}} h(t,x) &\geq C\big(E[v] - E[h](t)\big) \\ E[v] - E[h](t) &> C\big(E[h_0] - E[h](t)\big) \\ &= C\int_0^t \int_{(0,1)} h^n |\partial_x^3 h|^{\alpha+1} \,\mathrm{d}x\,\mathrm{d}s \\ &\geq C\big(\min_{x\in\bar{\Omega}} h(t,x)\big)^{1-\varepsilon} \end{aligned}$$

- Conclusion: $\dfrac{d}{dt}(E[v] - E[h](t)) \geq C(E[v] - E[h](t))^{1-\varepsilon}$
- Gronwall gives: $E[h](t) \longrightarrow 0$ if $\min h(t,x) < \frac{1}{2}$ for all times

Further projects and questions

Droplets

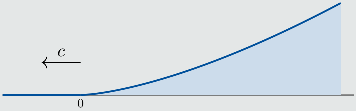
Study of droplet case

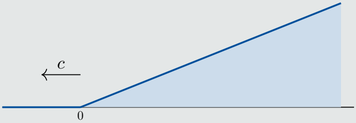


Question: Find solutions to full free-boundary problem

Travelling waves

Behaviour near contact points:





$h(t,x) = H(x - ct)$ and obtain ODE

$$cH = H^n |H'''|^{\alpha-1} H'''$$

Thank you for your attention!

Questions?