Long-time behaviour of solutions to non-Newtonian thin-film equations

joint work with Christina Lienstromberg (Stuttgart) and Katerina Nik (Vienna)

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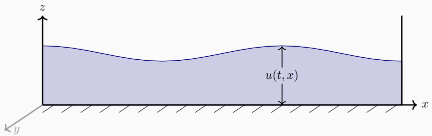
Non-Newtonian thin films

Thin fluid film

- incompressible
- viscous
- homogeneous in y-direction
- capillary-driven



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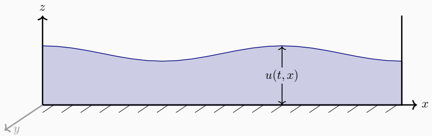


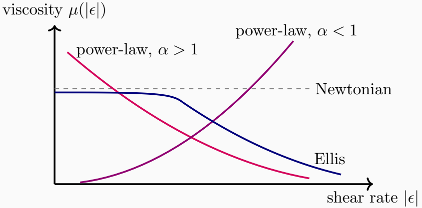
Non-Newtonian rheology

- power-law fluid:

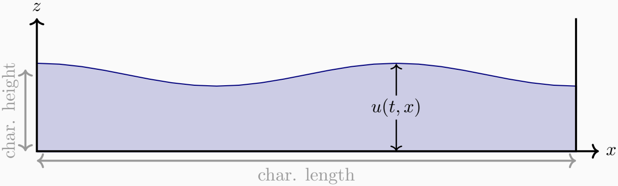
  $$\mu(|\epsilon|) = \mu_0 |\epsilon|^{\frac{1}{\alpha}-1}$$

- Ellis fluid:

  $$\frac{1}{\mu(|\epsilon|)} = \frac{1}{\mu_0}\left(1 + \left|\frac{\sigma(\epsilon)}{\sigma_{1/2}}\right|^{\alpha-1}\right)$$



Dynamics of non-Newtonian thin films



Lubrication approximation: $\frac{\text{char. height}}{\text{char. length}} \longrightarrow 0$

$$\begin{cases} \partial_t u + \partial_x\left(u^n |\partial_x^3 u|^{\alpha-1} \partial_x^3 u\right) &= 0, & t > 0, x \in \Omega \\ \partial_x u = u^n |\partial_x^3 u|^{\alpha-1} \partial_x^3 u &= 0, & t > 0, x \in \partial\Omega \\ u(0, x) &= u_0(x), & x \in \Omega \end{cases}$$

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Important properties

- Conservation of mass

$$\frac{d}{dt}\int_\Omega u(t,x)\mathrm{d}x = 0$$

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Important properties

- Conservation of mass
- Energy-dissipation inequality

$$E[u](t) + \int_0^t D[u](t)\mathrm{d}t = E[u_0]$$

$$E[u] = \frac{1}{2}\int_\Omega |\partial_x u|^2 \mathrm{d}x, \quad D[u] = \int_\Omega u^n|\partial_x^3 u|^{\alpha+1}\mathrm{d}x$$

Goal of this talk

Observation

The positive steady states are precisely the constant solutions.

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Questions

- If $u_0 \approx \bar{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 dx$, does then $u(t,x) \to \bar{u}_0$ as $t \to \infty$?
- Are there explicit bounds for the decay?

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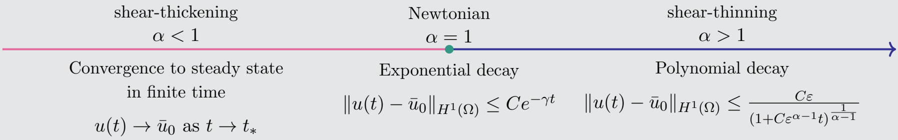
Answers

For all flow-behaviour exponents $\alpha > 0$ and all $\bar{u}_0$, there is $\varepsilon > 0$ such that $\|u_0 - \bar{u}_0\|_{H^1} < \varepsilon$ there is a global positive weak solution u such that



Short-time existence of weak solutions

Weak solutions

$$\int_0^T \langle u_t, \varphi \rangle_{W^1_{\alpha+1}(\Omega)} \, dt = \int_0^T \int_\Omega u^n \psi(\partial_x^3 u) \, \varphi_x \, dx \, dt, \quad \forall \varphi \in L_t^{\alpha+1} W_x^{1,\alpha+1}$$

here: $\psi(s) = |s|^{\alpha-1} s$

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Short history:

- Newtonian fluid $\alpha = 1$:
  - local positive strong solutions e.g. via semigroup theory
  - global non-negative weak solutions (Bernis–Friedman, Beretta–Bertsch–dalPasso, Bertozzi–Pugh,...)
- Non-Newtonian fluids
  - global non-negative weak solutions (Ansini–Giacomelli, Gladbach–J.–Lienstromberg)

Short-time existence of weak solutions

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We only need positive local weak solutions!

Construction

- Approximate ψ by $\psi_\sigma(s) = \left(s^2 + \sigma^2\right)^{\frac{\alpha-1}{2}} s$
- Construct solutions via semigroup theory
- uniform bounds and passage to the limit
- energy-dissipation equality
- bootstrap solutions as long as they remain positive

On the proof

Lojasiewicz–Simon-type inequality

$$\frac{d}{dt}E[u](t) = -D[u](t) \leq -C\big(E[u](t)\big)^{\frac{\alpha+1}{2}}$$

Strategy of the proof

- bootstrap local weak solutions
- apply Gronwall's inequality

$$\frac{2}{1-\alpha}\frac{d}{dt}\big(E[u](t)\big)^{\frac{1-\alpha}{2}} \leq -C$$

Further results

Ellis-law rheology

Dynamics of thin film

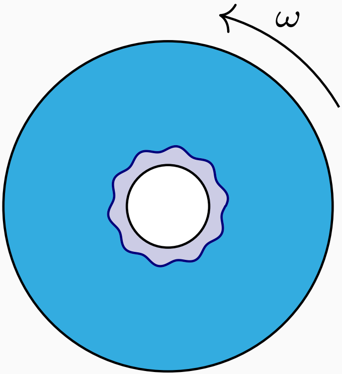
$$\begin{cases} u_t + \left(u^n(1+|uu_{xxx}|^{\alpha-1})u_{xxx}\right)_x = 0, & t>0,\ x\in\Omega, \\ u_x(t,x) = u_{xxx}(t,x) = 0, & t>0,\ x\in\partial\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \end{cases}$$

Long-time behaviour: $\|u(t,x) - \bar{u}_0\|_{H^1(\Omega)} \leq Ce^{-\gamma t}$

Further results

Different geometries

The same results hold in Taylor–Couette geometry
(Lienstromberg–Pernas-Castano–Velazquez, Lienstromberg–Velazquez)



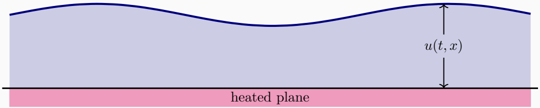
Further results

Guaranteed lift-off

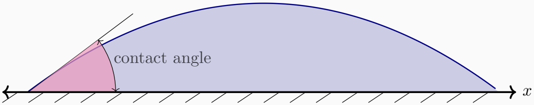
Small energy initial values have guaranteed lift-off in finite time.

Open problems and closing remarks

- Does similar long-time behaviour persist in different geometries or under inclusion of additional effects?



- Can one derive stability in the droplet case (orbital stability)?



Thank you for your attention!

Questions?