Pattern Formation and Film Rupture in an Asymptotic Model of the Bénard–Marangoni Problem

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joint work with Gabriele Brüll and Bastian Hilder

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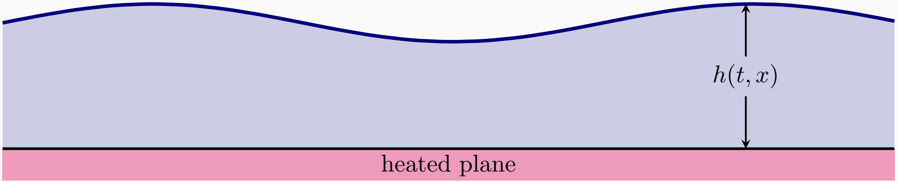
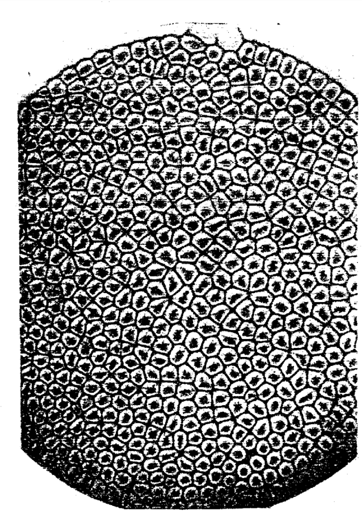
Thin fluid films on heated planes

Consider an incompressible Newtonian thin fluid film on a heated plane



Pattern formation in experiments

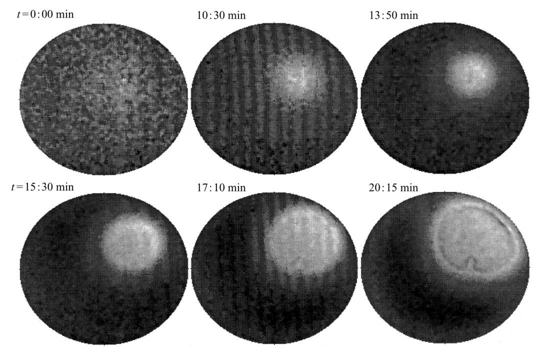


Source: H. Bénard, Les tourbillons cellulaires dans une nappe liquide, 1900.

In 1900,
Henri Bénard observed the formation of
regular polygonal pattern...

Pattern formation in experiments

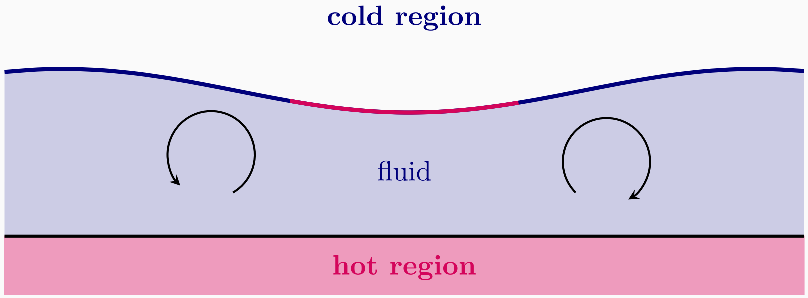
… and even
dewetting phenomena
have been demonstrated
experimentally.



Source: VanHook et al., Long-wavelength surface-tension-driven Bénard convection: experiment and theory, 1997.

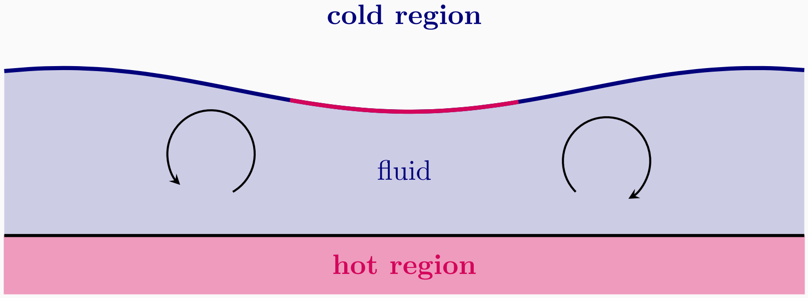
The thermocapillary effect

In 1956, M. J. Block explained the formation of pattern with the **thermocapillary effect**,
in 1958, J. R. A. Pearson confirmed this by theoretical considerations.



The thermocapillary effect

Due to dependence of surface tension on temperature, temperature variations lead to stress imbalances on the surface.



The Boussinesq–Navier–Stokes model

$$\begin{cases} \partial_t h + u_1 \partial_x h &= u_2 \\ \Sigma(p, \vec{u}) \cdot \vec{n} &= \sigma \kappa \vec{n} - (\partial_x \sigma) \vec{\tau} \\ \nabla T \cdot \vec{n} &= -K(T - T_g) \end{cases}$$

$$\begin{cases} \partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} &= -\nabla p + \mu \Delta \vec{u} - \vec{g} \\ \operatorname{div} \vec{u} &= 0 \\ \partial_t T + \vec{u} \cdot \nabla T &= \chi \Delta T \end{cases}$$

$$\begin{cases} \vec{u} &= 0 \\ T &= T_s \end{cases}$$

where $\sigma(x) = \sigma_0 - \alpha T(x)$

The asymptotic model

Rescaling $t \to \varepsilon^2 t$ and $x \to \varepsilon x$, gives dynamics for h in lowest order

Deformational model

$$\partial_t h + \partial_x \left[h^3(\partial_x^3 h - g\partial_x h) + M\frac{h^2}{(1+h)^2}\partial_x h \right] = 0, \quad t > 0,\ x \in \mathbb{R}$$

$g > 0$ gravitational constant

$M > 0$ Marangoni number $\sim T_s - T_g$

The dispersion relation

$$\partial_t h + \partial_x \left[h^3(\partial_x^3 h - g\partial_x h) + M\frac{h^2}{(1+h)^2}\partial_x h \right] = 0$$

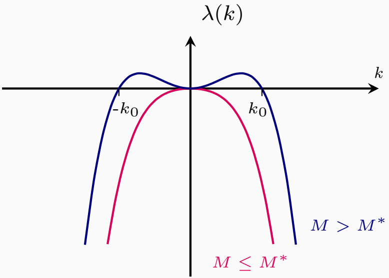
Linearisation about $\bar{h} = 1$:

$$\partial_t v = -\partial_x^4 v - \left(\frac{M}{4} - g\right)\partial_x^2 v$$

$v = \exp(\lambda t - ikx)$ is a solution iff

$$\lambda(k) = -k^4 + \left(\frac{M}{4} - g\right)k^2$$

(conserved) long-wave instability



Goal

study

bifurcation of stationary periodic pattern

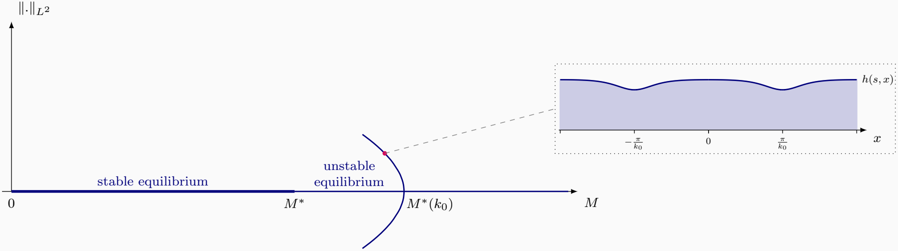
with fixed wave number k_0 at $M^*(k_0) = M^* + 4k_0^2$
and the existence of

film-rupture solutions

via global bifurcation theory

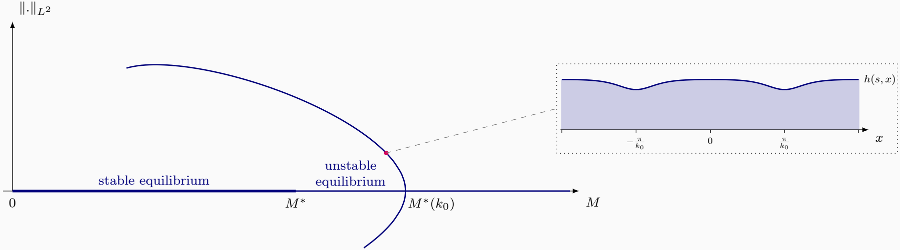
The main result

1. existence of a **local bifurcation branch** at $M^*(k_0) = M^* + 4k_0^2$ consisting of $\frac{2\pi}{k_0}$-periodic even stationary solutions



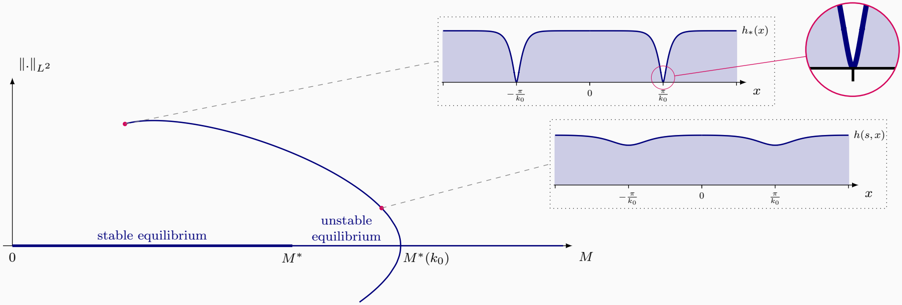
The main result

2. this branch can be extended to a **global bifurcation branch**

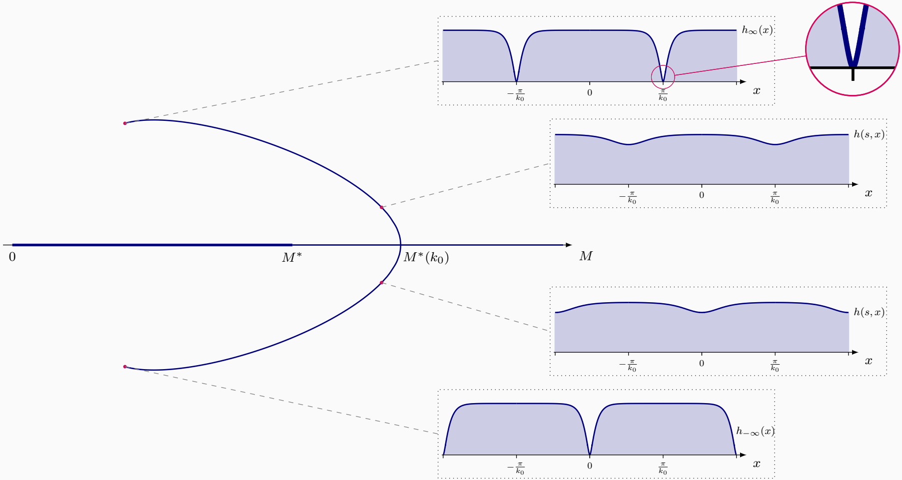


The main result

3. limit points exhibit **film rupture** and are weak even periodic stationary solutions



The main result



The main result

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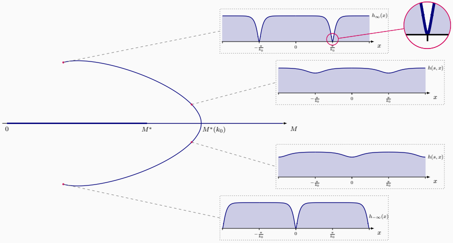

The bifurcation problem

$$\partial_x \left[h^3(\partial_x^3 h - g\partial_x h) + M\frac{h^2}{(1+h)^2}\partial_x h \right] = 0$$

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$$\partial_x^2 h - gh + M\left(\frac{1}{1+h} + \log\left(\frac{h}{1+h}\right)\right) + MK = 0$$

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Bifurcation problem for $h = 1 + v$

$$\mathcal{F}(v; M) = \partial_x^2 v - gv + M\left(\tfrac{1}{2+v} + \log\left(\tfrac{1+v}{2+v}\right)\right) + MK(v) = 0$$

where

$$K(v) = \fint_{-\pi/k_0}^{\pi/k_0} \tfrac{1}{2+v} + \log\left(\tfrac{1+v}{2+v}\right) \mathrm{d}x$$

function spaces

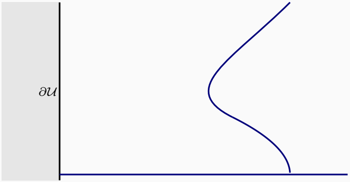
$$\mathcal{Y} = \left\{v \in L^2_{\mathrm{per}} : \fint v = 0,\ v \text{ even}\right\}, \quad \mathcal{X} = \mathcal{Y} \cap H^2_{\mathrm{per}}$$

$$\mathcal{U} = \{v \in \mathcal{X} : v > -1\}$$

Analytic global bifurcation theory

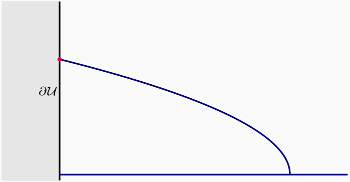
(C1) **blow-up**

$\|(v(s), M(s))\| \longrightarrow +\infty$



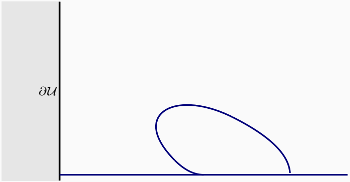
(C2) **film rupture**

$v(s) \longrightarrow \partial\mathcal{U}$



(C3) **closed loop**

$(v(s), M(s))$ is a closed loop



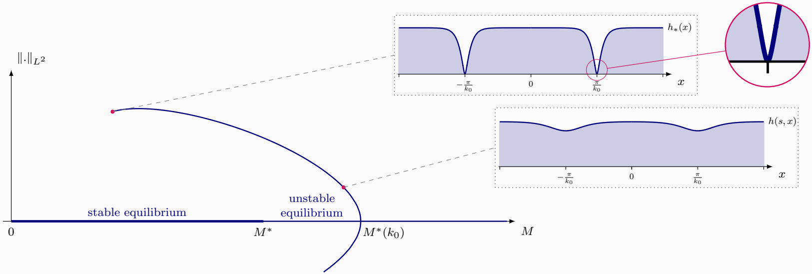
Establishing film rupture

Step 1: Ruling out

(C3) **closed loop**

$(v(s), M(s))$ is a closed loop

Nodal property

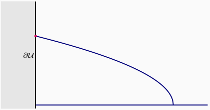


Establishing film rupture

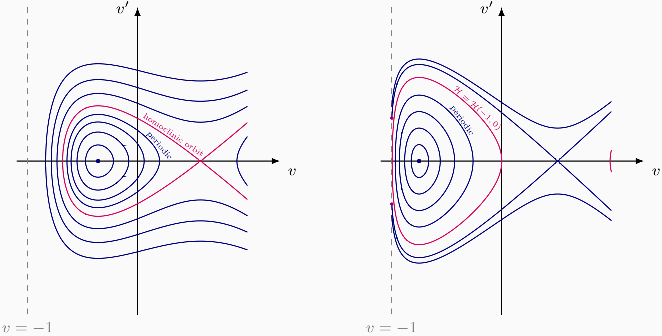
Step 2: showing that only

(C2) **film rupture**

$v(s) \longrightarrow \partial\mathcal{U}$



$\partial_x^2 v - gv + M\left(\frac{1}{2+v} + \log\left(\frac{1+v}{2+v}\right)\right) + MK = 0$ is a Hamiltonian system



Thank you for your attention!

Questions?

More information:

Manuscript of Talk

Publication